

On certain higher dimensional analogues of vertex algebras

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Abstract

A higher dimensional analogue of the notion of vertex algebra is formulated in terms of formal variable language with Borcherds' notion of G -vertex algebra as a motivation. Some examples are given and certain analogous duality properties are proved. Furthermore, it is proved that for any vector space W , any set of mutually local multi-variable vertex operators on W in a certain canonical way generates a vertex algebra with W as a natural module.

1 Introduction

Vertex (operator) algebras, introduced in mathematics ([B1], [FLM]), are known essentially to be chiral algebras of two dimensional conformal field theories, as formulated in physics ([BPZ], [MS]). In [B2], higher dimensional analogues of vertex algebras were established in terms of what were called relaxed multi-linear categories. (In fact, the setting of [B2] is extremely general.) It was proved in [Sn] (cf. [B2]) that this notion in a special case is equivalent to the notion of vertex algebra defined in [B1] and [FLM] in terms of formal variables. It is hoped that these algebras have the same relation to higher dimensional quantum field theories that vertex algebras have to one dimensional quantum field theories (or to “chiral halves” of two dimensional conformal field theories).

In vertex algebra theory, even though the notion of ordinary vertex algebra can be formulated in terms of \mathcal{D} -modules ([BD], [HL]) and in terms of relaxed multi-linear categories ([B2], [Sn]), the formal variable approach plays a unique role. Analogously, a formal variable formulation of higher dimensional analogues of vertex algebras should be of importance. The main purpose of this paper is to formulate certain higher dimensional analogues of vertex algebras in terms of formal variable language in the same way that the ordinary vertex algebras were defined in [B1] and [FLM].

Vertex algebras are often thought of as a kind of generalized algebras equipped with infinitely many bilinear operations. Another point of view is that vertex operator algebras are “algebras” of vertex operators just as classical (associative, or Lie) algebras are algebras of linear operators, where vertex operators on a vector space W are understood as elements of the space $\text{Hom}(W, W((x)))$. (This particular viewpoint was emphasized in

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[FLM].) Then for a vertex algebra V , each element v is represented by the vertex operator $Y(v, x)$ on V , where $Y(v, x)$ is an analogue of the left multiplication associated to an element for a classical non-associative algebra. In the definition of the notion of vertex algebra, the main axiom is what was called in [FLM] and [FHL] the Jacobi identity:

$$\begin{aligned} & z^{-1}\delta\left(\frac{x-y}{z}\right)Y(u, x)Y(v, y)w - z^{-1}\delta\left(\frac{-y+x}{z}\right)Y(v, y)Y(u, x)w \\ &= y^{-1}\delta\left(\frac{x-z}{y}\right)Y(Y(u, z)v, y)w. \end{aligned} \quad (1.1)$$

For higher dimensions, we again use the viewpoint that vertex (operator) algebras are “algebras” of local *multi-variable* vertex operators. Given a vector space W , we naturally consider elements of the space

$$\text{Hom}\left(W, W[[x_1, \dots, x_n]][x_1^{-1}, \dots, x_n^{-1}]\right) \quad (1.2)$$

as our n -variable vertex operators. For an n -dimensional analogue V of vertex algebras, each element v of V should be represented by an n -variable vertex operator $Y(v, \mathbf{x})$ on V , where $\mathbf{x} = (x_1, \dots, x_n)$. As in the ordinary case [DL] (cf. [FHL], [FLM], [G]), vertex operators $Y(u, \mathbf{x})$ and $Y(v, \mathbf{y})$ are assumed to be mutually local in the sense that

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x})Y(v, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y})Y(u, \mathbf{x}) \quad (1.3)$$

for some $k \in \mathbb{N}$. (See Section 2 for notations.) Note that in the 1-dimensional case, product $Y(u, x)Y(v, y)$ is described by rational functions with only three possible poles at $x = 0, \infty, y$ and the Jacobi identity is a version of classical Cauchy residue theorem ([FLM], [FHL]). In the n -dimensional case, from locality product $Y(u, \mathbf{x})Y(v, \mathbf{y})$ can be described by rational functions with only 3^n possible poles. But, there are still three *explicit* relevant quantities (two products and one iterate). What we do next is to employ formal Laurent series expansions in certain domains. For each variable x_i , we have two disjoint domains $|x_i| > |y_i| > 0$ and $|y_i| > |x_i| > 0$. This gives us total 2^n different domains for $\mathbf{x} = (x_1, \dots, x_n)$. Each such domain can be represented by a binary code $\alpha \in (F_2)^n$ of length n in the obvious way. For $\mathbf{m} \in \mathbf{Z}^n$, we then define 2^n formal series $\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}Y(u, \mathbf{x})Y(v, \mathbf{y}))$ which roughly speaking are the analytic continuation of $(\mathbf{x} - \mathbf{y})^{\mathbf{m}}Y(u, \mathbf{x})Y(v, \mathbf{y})$ in the 2^n different domains. Note that the products $(\mathbf{x} - \mathbf{y})^{\mathbf{m}}Y(u, \mathbf{x})Y(v, \mathbf{y})$ and $(-\mathbf{y} + \mathbf{x})^{\mathbf{m}}Y(v, \mathbf{y})Y(u, \mathbf{x})$ are two special cases. So, each $\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}Y(u, \mathbf{x})Y(v, \mathbf{y}))$ can be considered as an implicit product. Now our multi-variable Jacobi identity reads as

$$\begin{aligned} & \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta\left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}}\right) Y(u, \mathbf{x})Y(v, \mathbf{y}) \right) \\ &= \mathbf{y}^{-1} \delta\left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}}\right) Y(Y(u, \mathbf{z})v, \mathbf{y}), \end{aligned} \quad (1.4)$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are n -dimensional variables and the delta functions are n -dimensional. In the 1-dimensional case, Jacobi identity does not require locality, instead, it implies locality. For higher dimensional cases, to state the Jacobi identity, or for $\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^m Y(u, \mathbf{x}) Y(v, \mathbf{y}))$ to be defined, we need the locality of $Y(u, \mathbf{x})$ and $Y(v, \mathbf{y})$ as a prerequisite. Now, the main axioms of our higher dimensional analogue of the notion of vertex algebra are locality and the above Jacobi identity. Typical examples of such vertex algebras are tensor products of n ordinary vertex algebras.

Once this higher dimensional analogue of the notion of vertex algebra is defined, all the results in the ordinary case (see [FLM], [FHL], [DL], [Li2]) are carried over, and what is more, the same old proofs work fine almost all the time. Especially, it is proved that locality (or weak commutativity) together with some minor information is strong enough to recover the Jacobi identity. Thus, one may define the notion of vertex algebra by using locality instead of the Jacobi identity as the main axiom (see [DL], [FHL], [FLM], [G], [Li2], see also [Ka]). This is analogous to a phenomenon in the classical commutative associative algebra theory. It is known that vertex algebras are analogous to commutativity associative algebras with identity while locality is analogous to commutativity for left multiplications associated with elements of classical algebras. As it was explained in [Li2], commutativity associative algebras with identity are exactly non-associative algebras with identity with all left multiplications being commuting. However, in defining the notion of module for a commutative associative algebra we use *associativity* instead of *commutativity*. It seems that it is a universal principle that the notion of module should be defined by using all the axioms that make sense in defining the notion of the algebra. In view of this, using Jacobi identity to define the notion of vertex algebra is more natural than using locality.

As a good practice of the viewpoint that vertex (operator) algebras are “algebras” of vertex operators, in [Li2] we established an analogue of endomorphism algebra. It was proved that any maximal space of pair-wise mutually local vertex operators on a vector space W has a canonical vertex algebra structure with W as a natural module. In terms of vertex algebras of local vertex operators, the notion of module and the notion of representation are identified just as in classical theories. Furthermore, it follows from Zorn’s Lemma that for an abstract vector space W , any set of mutually local vertex operators on W in a certain canonical way generates a vertex algebra (a subspace of $\text{Hom}(W, W((x)))$) with W as a natural module. As mentioned in [Li2], this result is an analogue of the trivial classical fact — any set of commuting linear operators on a vector space U generates a commutative associative algebra with U as a module. In addition to its conceptual importance, this result is quite useful to construct examples of vertex algebras and their modules.

Certain results similar to that of [Li2] were also independently obtained in [LZ] and the work [LZ] is also in this spirit.

In this paper, we establish the main results of [Li2] for higher dimensional analogues of vertex algebras. In fact, this paper is modeled on [Li2]. In the 1-dimensional case, the adjoint vertex operator (vertex operators on the space of vertex operators) was defined in terms of commutators (or more generally cross brackets) and normal-order products. For higher dimensions, normal-order product is not defined in general. Note that for mutually local vertex operators ψ, ϕ and for $\mathbf{m} \in \mathbf{Z}^n$, even though

$$(\psi_{[\mathbf{m}]}\phi)(\mathbf{y}) = \text{Res}_{\mathbf{x}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}\psi(\mathbf{x})\phi(\mathbf{y}) - (-\mathbf{y} + \mathbf{x})^{\mathbf{m}}\phi(\mathbf{y})\psi(\mathbf{x}))$$

defines a vertex operator, it is not what we want. (In general, $\psi_{[\mathbf{m}]}\phi$ may not be local with the old vertex operators ψ and ϕ .) Here, we use an analogue of the Jacobi identity (or just the iterate formula) to define the adjoint vertex operators. Then the main results of [Li2] for higher dimensions are carried over.

This paper is motivated by [B2] and presumably, the n dimensional analogue formulated in this paper is equivalent to Borcherds' notion in [B2] of commutative G_n -vertex algebra, where $G_n = (H_n, K_n)$ is what was called therein an elementary vertex group with

$$H_n = \mathbf{C}[D_1, \dots, D_n], \tag{1.5}$$

$$K_n = \mathbf{C}[[x_1, \dots, x_n]][x_1^{-1}, \dots, x_n^{-1}]. \tag{1.6}$$

(See Remark 2.5 for more information.) Most likely, Snydal's proof in [Sn] will extend to higher dimensions with some changes. In view of this, our n dimensional analogues of vertex algebras are named as G_n -vertex algebras and the vertex operators we consider are named as G_n -vertex operators..

In [KO], a certain notion of vertex algebra (with $n = 2$) was introduced with an axiom in terms of operator product expansions. As mentioned in [KO], if we only allow integral powers of formal variables in the definition of the notion of vertex algebra in [KO], locality follows from the axioms. Then as mentioned before, the Jacobi identity will follow. Therefore, the notion of vertex algebra defined in [KO] in the special case is equivalent to the notion of G_2 -vertex algebra defined in this paper. On the other hand, we emphasize that our definition of vertex algebra naturally give rise to the (right) definition of module and that the main axiom, Jacobi identity, is crucial in constructing vertex algebras from given vertex operators.

In the ordinary case, among the important and interesting vertex (operator) algebras are those constructed from certain highest weight representations of affine Lie algebras (see [FZ]). Then one naturally wants to consider the multi-variable generalizations of affine Lie algebras — toroidal Lie algebras (cf. [BBS]). For toroidal Lie algebras, one can form generating functions in n -variables and from [BBS], one can easily check that all generating series are pair-wise mutually local. However, on the modules W constructed in [BBS], generating series $a(\mathbf{x})$ for $a \in \mathfrak{g}$ do not give rise to G_n -vertex operators on W

because $a(\mathbf{x})v$ in general involves infinitely many negative powers of variables x_2, \dots, x_n . So, our results here cannot apply to toroidal Lie algebras. We believe that it is possible to relate this type of modules for toroidal Lie algebras to Borcherds' G -vertex algebras of a certain type. This is currently under investigation in [Li3].

This paper is organized as follows: In Section 2, we define certain generalized iota-maps and present some basic properties. In Section 3, we define higher-dimensional analogues of vertex algebras and present basic duality properties. In Section 4, we present the theory of vertex algebras of multi-variable vertex operators.

2 Basics of formal calculus and generalized iota-maps

In this section, we first collect basic notations and conventions in formal calculus and we then define G_n -vertex operators and certain generalized iota-maps. We also present some basic properties of the generalized iota-maps.

We here mainly follow [FLM] and [FHL] for the treatment of formal calculus. Throughout this paper, $x, x_0, x_1, x_2, \dots, y, y_0, y_1, y_2, \dots, z, z_0, z_1, z_2, \dots$ are independent commuting formal variables. We typically use U for a general vector space in establishing certain definitions and notations.

Let n be a positive integer fixed throughout this paper. Set

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n), \quad \mathbf{z} = (z_1, \dots, z_n). \quad (2.1)$$

For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$, we set

$$\mathbf{x}^\mathbf{m} = x_1^{m_1} \cdots x_n^{m_n}. \quad (2.2)$$

For $k \in \mathbf{Z}$, set

$$\mathbf{x}^k = x_1^k \cdots x_n^k. \quad (2.3)$$

Then, for $\mathbf{m} \in \mathbf{Z}^n$, $k \in \mathbf{Z}$, we write $\mathbf{x}^{\mathbf{m}+k}$ for $\mathbf{x}^\mathbf{m}\mathbf{x}^k$. In particular, $\mathbf{x}^{-\mathbf{m}-1}$ is defined this way.

For a vector space U , denote by $U[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$ the space of all formal integral power series in x_1, \dots, x_n with coefficients in U . That is,

$$U[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]] = \left\{ \sum_{\mathbf{m} \in \mathbf{Z}^n} f(\mathbf{m}) \mathbf{x}^\mathbf{m} \mid f(\mathbf{m}) \in U \quad \text{for } \mathbf{m} \in \mathbf{Z}^n \right\}. \quad (2.4)$$

Denote by $U[[x_1, \dots, x_n]]$ the subspace of all formal nonnegative power series and by $U((x_1, \dots, x_n))$ the subspace of all formal Laurent (lower truncated) series, i.e.,

$$U((x_1, \dots, x_n)) = U[[x_1, \dots, x_n]][x_1^{-1}, \dots, x_n^{-1}]. \quad (2.5)$$

We shall also use the notations $U[[\mathbf{x}]]$ and $U((\mathbf{x}))$ for $U[[x_1, \dots, x_n]]$ and $U((x_1, \dots, x_n))$, respectively.

In formal calculus, we shall heavily use the following binomial expansion convention:

$$(x - y)^m = \sum_{i \geq 0} \binom{m}{i} (-1)^i x^{m-i} y^i \in \mathbf{C}[x, x^{-1}][[y]] \quad (2.6)$$

for $m \in \mathbf{Z}$, where $\binom{m}{i} = \frac{1}{i!} m(m-1) \cdots (m-i+1)$.

For $k \in \mathbf{Z}$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$, we define

$$(\mathbf{x} - \mathbf{y})^k = (x_1 - y_1)^{k_1} \cdots (x_n - y_n)^{k_n}, \quad (2.7)$$

$$(\mathbf{x} - \mathbf{y})^{\mathbf{m}} = (x_1 - y_1)^{m_1} \cdots (x_n - y_n)^{m_n}. \quad (2.8)$$

Then, for $\mathbf{m} \in \mathbf{Z}^n$, $k \in \mathbf{Z}$, we define $(\mathbf{x} - \mathbf{y})^{\mathbf{m}+k}$ to be $(\mathbf{x} - \mathbf{y})^{\mathbf{m}}(\mathbf{x} - \mathbf{y})^k$. For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$ and $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{N}^n$, we set

$$\binom{\mathbf{m}}{\mathbf{i}} = \binom{m_1}{i_1} \cdots \binom{m_n}{i_n}. \quad (2.9)$$

Then

$$(\mathbf{x} + \mathbf{y})^{\mathbf{m}} = \sum_{\mathbf{i} \in \mathbf{N}^n} \binom{\mathbf{m}}{\mathbf{i}} \mathbf{x}^{\mathbf{m}-\mathbf{i}} \mathbf{y}^{\mathbf{i}}. \quad (2.10)$$

Recall from [FLM] that for $A(\mathbf{x}), B(\mathbf{x}), C(\mathbf{x}) \in (\text{End } U)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$, the following associativity

$$A(\mathbf{x})(B(\mathbf{x})C(\mathbf{x})) = (A(\mathbf{x})B(\mathbf{x}))C(\mathbf{x}) \quad (2.11)$$

holds *provided* that all the products $A(\mathbf{x})B(\mathbf{x})C(\mathbf{x})$, $A(\mathbf{x})B(\mathbf{x})$ and $B(\mathbf{x})C(\mathbf{x})$ exist. The assumption of the existence of the products is crucial. As it was illustrated in [FLM], abuse of this associativity could lead to paradoxes. For convenience, we refer the *associativity assumption* for the product $A(\mathbf{x})B(\mathbf{x})C(\mathbf{x})$ to the existence of the three products.

Remark 2.1 We have the following cancelation law. Let

$$f(\mathbf{x}, \mathbf{y}), g(\mathbf{x}, \mathbf{y}) \in U[[x_i, x_i^{-1}, y_i, y_i^{-1} \mid i = 1, \dots, n]]$$

be such that $f(\mathbf{x}, \mathbf{y})$ and $g(\mathbf{x}, \mathbf{y})$ involve only finitely many negative powers of x_i for $i \in I$, where I is a subset of $\{1, \dots, n\}$. Suppose that there exist nonnegative integers k_i for $i \in I$ such that

$$\left(\coprod_{i \in I} (x_i - y_i)^{k_i} \right) f(\mathbf{x}, \mathbf{y}) = \left(\coprod_{i \in I} (x_i - y_i)^{k_i} \right) g(\mathbf{x}, \mathbf{y}). \quad (2.12)$$

Then $f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y})$. Indeed, under the given assumption on $f(\mathbf{x}, \mathbf{y})$ and $g(\mathbf{x}, \mathbf{y})$, the associativity assumption holds for the products

$$\left(\coprod_{i \in I} (-y_i + x_i)^{-k_i} \right) \left(\coprod_{i \in I} (x_i - y_i)^{k_i} \right) f(\mathbf{x}, \mathbf{y})$$

and

$$\left(\coprod_{i \in I} (-y_i + x_i)^{-k_i} \right) \left(\coprod_{i \in I} (x_i - y_i)^{k_i} \right) g(\mathbf{x}, \mathbf{y}).$$

Then multiplying both sides of (2.12) by $\coprod_{i \in I} (-y_i + x_i)^{-k_i}$ and using associativity we obtain $f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y})$.

The formal residue $\text{Res}_x f(x)$ of a formal series $f(x) = \sum_{n \in \mathbf{Z}} f(n)x^n \in U[[x, x^{-1}]]$ is defined to be the coefficient $f(-1)$ of x^{-1} in $f(x)$. We shall use the following convention:

$$\text{Res}_{\mathbf{x}} = \text{Res}_{x_1} \cdots \text{Res}_{x_n}. \quad (2.13)$$

The formal delta function is defined to be a formal series:

$$\delta(x) = \sum_{m \in \mathbf{Z}} x^m \in \mathbf{C}[[x, x^{-1}]]. \quad (2.14)$$

It is a so-called expansion of zero (see [FLM]):

$$\delta(x) = (1 - x)^{-1} - (-x + 1)^{-1} \quad (2.15)$$

in terms of the binomial expansion convention. Basic properties of delta function are:

$$f(z)\delta(z) = f(1)\delta(z) \quad \text{for } f(z) \in \mathbf{C}[z, z^{-1}], \quad (2.16)$$

$$g(z_1, z_2)z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) = g(z_1, z_1)z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) \quad (2.17)$$

for $g(z_1, z_2) \in \mathbf{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$ such that $g(z_1, z_1)$ exists (algebraically), e.g., $g(z_1, z_2) \in \mathbf{C}((z_1, z_2))$. Furthermore,

$$h(z_0, z_1, z_2)z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) = h(z_1 - z_2, z_1, z_2)z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) \quad (2.18)$$

for $h(z_0, z_1, z_2) \in U((z_0, z_1, z_2))$.

The following are the fundamental delta function identities in the formal variable approach to vertex (operator) algebras:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) = z_1^{-1}\delta\left(\frac{z_0 + z_2}{z_1}\right), \quad (2.19)$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right). \quad (2.20)$$

We define the following multi-variable delta functions:

$$\delta(\mathbf{x}) = \delta(x_1) \cdots \delta(x_n), \quad (2.21)$$

$$\mathbf{y}^{-1}\delta\left(\frac{\mathbf{x}}{\mathbf{y}}\right) = y_1^{-1}\delta\left(\frac{x_1}{y_1}\right) \cdots y_n^{-1}\delta\left(\frac{x_n}{y_n}\right), \quad (2.22)$$

$$\mathbf{z}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{z}}\right) = z_1^{-1}\delta\left(\frac{x_1-y_1}{z_1}\right) \cdots z_n^{-1}\delta\left(\frac{x_n-y_n}{z_n}\right). \quad (2.23)$$

Then using (2.19) we have

$$\mathbf{z}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{z}}\right) = \mathbf{x}^{-1}\delta\left(\frac{\mathbf{z}+\mathbf{y}}{\mathbf{x}}\right). \quad (2.24)$$

Furthermore, using (2.18) we get

$$\mathbf{z}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{z}}\right) f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{z}}\right) f(\mathbf{x}, \mathbf{y}, \mathbf{x}-\mathbf{y}) \quad (2.25)$$

for $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in U((\mathbf{x}, \mathbf{y}, \mathbf{z}))$.

Now we discuss a multi-dimensional analogue of (2.20). First, (2.20) immediately gives

$$\prod_{i=1}^n \left(z_i^{-1}\delta\left(\frac{x_i-y_i}{z_i}\right) - z_i^{-1}\delta\left(\frac{-y_i+x_i}{z_i}\right) \right) = \prod_{i=1}^n y_i^{-1}\delta\left(\frac{x_i-z_i}{y_i}\right) = \mathbf{y}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{z}}{\mathbf{y}}\right). \quad (2.26)$$

To express the left-hand side in terms of n -dimensional delta functions, we need to expand it into 2^n -terms.

Let $F_2 = \{0, 1\}$ be the 2-element field. Then $(F_2)^n$ is an n -dimensional vector space over F_2 . For $\alpha \in (F_2)^n$, denote by $|\alpha|$ the weight of α , which is the number of nonzero α_i 's in the expression $\alpha = (\alpha_1, \dots, \alpha_n)$. Denote by e_i the element of $(F_2)^n$ all of whose coordinates are 0 except the i th coordinate which is 1. For $1 \leq i \leq n$, we define a translation σ_i on $(F_2)^n$ by

$$\sigma_i(\alpha) = \alpha + e_i \quad \text{for } \alpha \in (F_2)^n. \quad (2.27)$$

Then $\sigma_i^2 = 1$ and

$$(-1)^{|\sigma_i(\alpha)|} = -(-1)^{|\alpha|}. \quad (2.28)$$

Set

$$E_n = \{\alpha \in (F_2)^n \mid |\alpha| \in 2\mathbf{Z}\}. \quad (2.29)$$

Then

$$(F_2)^n = E_n \cup \sigma_i(E_n) \quad (\text{disjoint union}) \quad (2.30)$$

for $i = 1, \dots, n$.

In the following we shall extend the iota-maps defined in [FLM] and [FHL] and in [Le] to define linear maps $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$ for $\alpha \in (F_2)^n$. Their common domain will be the following subalgebra of the field of fractions of the ring $\mathbf{C}[[\mathbf{x}, \mathbf{y}]]$:

$$A = \mathbf{C}[[\mathbf{x}, \mathbf{y}]] [x_i^{-1}, y_i^{-1}, (x_i + y_i)^{-1}, (x_i - y_i)^{-1} \mid i = 1, \dots, n]. \quad (2.31)$$

For each $\alpha \in (F_2)^n$, we define a linear map

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}} : A \rightarrow \mathbf{C}[[x_i, x_i^{-1}, y_i, y_i^{-1} \mid i = 1, \dots, n]]$$

as follows: First, for $1 \leq i \leq n$, $m \in \mathbf{Z}$, we define

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((x_i \pm y_i)^m) = \begin{cases} (x_i \pm y_i)^m & \text{if } \alpha_i = 0 \\ (\pm y_i + x_i)^m & \text{if } \alpha_i = 1. \end{cases} \quad (2.32)$$

(We are using the binomial expansion convention.) Next, we define

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} \pm \mathbf{y})^{\mathbf{m}}) = \prod_{i=1}^n \iota_{\alpha, \mathbf{x}, \mathbf{y}}((x_i \pm y_i)^{m_i}) \quad (2.33)$$

for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$. Then we define

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x}^{\mathbf{r}} \mathbf{y}^{\mathbf{s}} (\mathbf{x} \pm \mathbf{y})^{\mathbf{m}} g(\mathbf{x}, \mathbf{y})) = \mathbf{x}^{\mathbf{r}} \mathbf{y}^{\mathbf{s}} g(\mathbf{x}, \mathbf{y}) \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} \pm \mathbf{y})^{\mathbf{m}}) \quad (2.34)$$

for $\mathbf{r}, \mathbf{s}, \mathbf{m} \in \mathbf{Z}^n$, $g(\mathbf{x}, \mathbf{y}) \in \mathbf{C}[[\mathbf{x}, \mathbf{y}]]$. It is clear that each map $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$ is linear over the Laurent polynomial algebra in x_i, y_i for $i = 1, \dots, n$ and that $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$ commutes with all the formal partial differential operators ∂_{x_i} and ∂_{y_i} .

Now, we can write (2.26) as:

$$\mathbf{y}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}} \right) = \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \right), \quad (2.35)$$

where $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$ is naturally extended on $A[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$. Furthermore, let $r, s, t \in \mathbf{Z}$ and $p(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in U((\mathbf{x}, \mathbf{y}, \mathbf{z}))$. Multiplying both sides of (2.35) by $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and then using the substitution rule (2.24) and (2.25) we get

$$\begin{aligned} & \mathbf{y}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}} \right) p(\mathbf{y} + \mathbf{z}, \mathbf{y}, \mathbf{z}) \\ &= \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) p(\mathbf{x}, \mathbf{y}, \mathbf{x} - \mathbf{y}) \right). \end{aligned} \quad (2.36)$$

Remark 2.2 Set

$$H_n = \mathbf{C}[D_1, \dots, D_n], \quad (2.37)$$

where D_i 's are independent commuting indeterminants or formal variables. Then H_n , being identified with the universal enveloping algebra of the abelian Lie algebra with a basis $\{D_1, \dots, D_n\}$, has a cocommutative Hopf algebra structure. The dual of H_n is a commutative associative algebra which can be canonically identified as

$$H_n^* = \mathbf{C}[[x_1, \dots, x_n]]. \quad (2.38)$$

Set

$$K_n = \mathbf{C}[[x_1, \dots, x_n]][x_1^{-1}, \dots, x_n^{-1}], \quad (2.39)$$

which is a natural commutative associative algebra and a natural H_n^* -module. Furthermore, K_n is an H_n -module with D_i acting as ∂_{x_i} for $i = 1, \dots, n$. Define G_n to be the pair (H_n, K_n) equipped with the above structures. From [B2], the pair G_n is what was called therein an elementary vertex group. In this paper, we shall at least superficially only use G_n as a symbol.

Definition 2.3 (cf. [B2]) Let W be a vector space. A G_n -vertex operator on W is a formal series

$$\psi(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbf{Z}^n} \psi(\mathbf{m}) \mathbf{x}^{-\mathbf{m}-1} \in (\text{End } W)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$$

such that $\psi(\mathbf{x})w \in W((\mathbf{x}))$ for $w \in W$.

All G_n -vertex operators on W form a subspace, which we denote by $VO_{G_n}(W)$.

Remark 2.4 In practice, we shall consider a G_n -vertex operator on W as an object such as a map from \mathbf{Z}^n to $\text{End } W$, which can be uniquely represented by a formal series $\psi(\mathbf{x})$ in *any* formal variable \mathbf{x} . In this way, we may use the notations ψ , $\psi(\mathbf{x})$ or $\psi(\mathbf{y})$ for the vertex operator.

Definition 2.5 G_n -vertex operators ψ and ϕ are said to be *mutually local* if there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x}) \phi(\mathbf{y}) = (\mathbf{x} - \mathbf{y})^k \psi(\mathbf{y}) \phi(\mathbf{x}). \quad (2.40)$$

A set S of G_n -vertex operators on W is said to be *local* if any two vertex operators (maybe the same) in S are mutually local.

Remark 2.6 Note that if (2.40) holds for a certain k , then it also holds for a bigger k .

Remark 2.7 In [KO], a notion of quantum field in one variable and two variables were introduced where arbitrary complex powers are allowed. Note that $VO_{G_2}(W) \subset QF_1(W)$ and $VO_{G_4}(W) \subset QF_2(W)$, where $QF_1(W)$ and $QF_2(W)$ are the space of quantum fields in one variable and two variables defined in [KO].

Let $\psi, \phi \in VO_{G_n}(W)$. For $w \in W$, $\psi(\mathbf{x})\phi(\mathbf{y})w$ involves only finitely many negative powers of all y_i 's, but it may involve infinitely many negative powers of x_i 's. Therefore, in general, the product $\psi(\mathbf{x})\phi(\mathbf{y})$ does not represent an element of $VO_{G_{2n}}(W)$.

Lemma 2.8 Let ψ and ϕ be mutually local G_n -vertex operators on W . Let $k \in \mathbb{N}$ be such that (2.40) holds. Then

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y}), \quad (\mathbf{x} - \mathbf{y})^k \phi(\mathbf{y})\psi(\mathbf{x}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y}))). \quad (2.41)$$

Proof. Let $w \in W$. Because $(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y})w$ involves only finitely many negative powers of y_i 's and $(\mathbf{x} - \mathbf{y})^k \phi(\mathbf{y})\psi(\mathbf{x})w$ involves only finitely many negative powers of x_i 's, by (2.40) we have

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y})w = (\mathbf{x} - \mathbf{y})^k \phi(\mathbf{y})\psi(\mathbf{x})w \in W((\mathbf{x}, \mathbf{y})). \quad (2.42)$$

This proves (2.41). \square

Remark 2.9 Let $W = \coprod_{\lambda \in \mathbf{C}^n} W(\lambda)$ be a \mathbf{C}^n -graded vector space satisfying that for any $\lambda \in \mathbf{C}^n$, $W(\mathbf{m} + \lambda) = 0$ for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$ with m_i being sufficiently small for some i . Let

$$\psi(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbf{Z}^n} \psi(\mathbf{m}) \mathbf{x}^{-\mathbf{m}-1}$$

be a *homogeneous* G_n -vertex operator of weight $h = (h_1, \dots, h_n) \in \mathbf{C}^n$ in the sense that

$$\psi(\mathbf{m})W(\lambda) \subset W(\lambda + h - \mathbf{m} - 1) \quad \text{for } \mathbf{m} \in \mathbf{Z}^n, \quad \lambda \in \mathbf{C}^n, \quad (2.43)$$

where we consider 1 as the element $(1, \dots, 1)$ of \mathbf{C}^n . Let $W' = \coprod_{\lambda \in \mathbf{C}^n} W(\lambda)^*$ be the restricted dual. Then

$$\langle w', \psi(\mathbf{x})w \rangle \in \mathbf{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] \quad (2.44)$$

for $w' \in W'$, $w \in W$. Furthermore, let ψ and ϕ be mutually local homogeneous G_n -vertex operators on W . Then the formal series

$$\langle w', \psi(\mathbf{x})\phi(\mathbf{y})w \rangle$$

absolutely converges in the domain $|x_1| > |y_1| > 0, \dots, |x_n| > |y_n| > 0$ to a rational function of the form

$$h(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}, \mathbf{y}) / (\mathbf{x}^{\mathbf{r}} \mathbf{y}^{\mathbf{s}} (\mathbf{x} - \mathbf{y})^{\mathbf{k}}) \quad (2.45)$$

where $g(\mathbf{x}, \mathbf{y}) \in \mathbf{C}[\mathbf{x}, \mathbf{y}]$ and $\mathbf{r}, \mathbf{s}, \mathbf{k} \in \mathbf{Z}^n$.

Let $R(W, \mathbf{x}, \mathbf{y})$ consist of formal series

$$f(\mathbf{x}, \mathbf{y}) \in (\text{End } W)[[x_i, x_i^{-1}, y_i, y_i^{-1} \mid i = 1, \dots, n]]$$

for which there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y}))), \quad (2.46)$$

i.e., $(\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \in VO_{2n}(W)$. Clearly, $R(W, \mathbf{x}, \mathbf{y})$ is a vector subspace. In view of Lemma 2.8, if ψ and ϕ are mutually local G_n -vertex operators on W , then

$$\psi(\mathbf{x})\phi(\mathbf{y}), \quad \phi(\mathbf{y})\psi(\mathbf{x}) \in R(W, \mathbf{x}, \mathbf{y}). \quad (2.47)$$

Remark 2.10 Note that for any $g(\mathbf{x}, \mathbf{y}) \in V_{G_{2n}}(W)$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}) g(\mathbf{x}, \mathbf{y}) \quad \text{exists} \quad (2.48)$$

in $(\text{End } W)[[x_i, x_i^{-1}, y_i, y_i^{-1} \mid i = 1, \dots, n]]$ for every $\alpha = (\alpha_1, \dots, \alpha_n) \in (F_2)^n$, $\mathbf{m} \in \mathbf{Z}^n$. Furthermore, for $w \in W$, $\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}) g(\mathbf{x}, \mathbf{y})w$ involves only finitely many negative powers of x_i and y_j for i, j with $\alpha_i = 1, \alpha_j = 0$ because $\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}})$ has this property.

Definition 2.11 For $\alpha \in (F_2)^n$, we define a linear map

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}} : R(W, \mathbf{x}, \mathbf{y}) \rightarrow (\text{End } W)[[x_i, x_i^{-1}, y_i, y_i^{-1} \mid i = 1, \dots, n]]$$

by

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y}))w = \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k}) ((\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y})w) \quad (2.49)$$

for $f(\mathbf{x}, \mathbf{y}) \in R(W, \mathbf{x}, \mathbf{y})$, $w \in W$, where k is a nonnegative integer such that (2.46) holds.

In view of Remark 2.10, the expression on the right-hand side of (2.49) exists. Furthermore, it does not depend on k . Indeed, let $k_1 > k_2$ be nonnegative integers such that (2.46) holds for $k = k_1, k_2$. Then

$$\begin{aligned} & \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k_1}) ((\mathbf{x} - \mathbf{y})^{k_1} f(\mathbf{x}, \mathbf{y})) \\ &= \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k_1}) (\mathbf{x} - \mathbf{y})^{k_1 - k_2} ((\mathbf{x} - \mathbf{y})^{k_2} f(\mathbf{x}, \mathbf{y})) \\ &= \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k_2}) ((\mathbf{x} - \mathbf{y})^{k_2} f(\mathbf{x}, \mathbf{y})). \end{aligned}$$

Remark 2.12 Note that in general, $\iota_{\alpha,\mathbf{x},\mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k})f(\mathbf{x}, \mathbf{y})$ may not exist. Thus all the parenthesis in (2.49) are absolutely necessary. In general, one should carefully use the associativity to multiply products.

Remark 2.13 Let ψ and ϕ be mutually local G_n -vertex operators on W . With $\psi(\mathbf{x})\phi(\mathbf{y})$ and $\phi(\mathbf{y})\psi(\mathbf{x})$ being in $R(W, \mathbf{x}, \mathbf{y})$, $\iota_{\alpha,\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))$ and $\iota_{\alpha,\mathbf{x},\mathbf{y}}(\phi(\mathbf{y})\psi(\mathbf{x}))$ are defined. Furthermore, from definition and the locality assumption we have

$$\iota_{\alpha,\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = \iota_{\alpha,\mathbf{x},\mathbf{y}}(\phi(\mathbf{y})\psi(\mathbf{x})) \quad (2.50)$$

for every $\alpha \in (F_2)^n$. Especially, we have

$$\iota_{\mathbf{0},\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = \psi(\mathbf{x})\phi(\mathbf{y}), \quad (2.51)$$

$$\iota_{\mathbf{1},\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = \phi(\mathbf{y})\psi(\mathbf{x}), \quad (2.52)$$

where $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

From the definition we immediately have:

Lemma 2.14 For $f(\mathbf{x}, \mathbf{y}) \in R(W, \mathbf{x}, \mathbf{y})$, there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k \iota_{\alpha,\mathbf{x},\mathbf{y}}(f(\mathbf{x}, \mathbf{y})) = (\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \quad (2.53)$$

for all $\alpha \in (F_2)^n$. In particular, we may take $k \geq 0$ such that

$$(\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y}))). \quad \square$$

Remark 2.15 It follows from (2.49) and Remark 2.10 that for $w \in W$, $\iota_{\alpha,\mathbf{x},\mathbf{y}}(f(\mathbf{x}, \mathbf{y}))w$ involves only finitely many negative (integral) powers of x_i and y_j with $\alpha_i = 1$ and $\alpha_j = 0$. For example, $\iota_{e_i,\mathbf{x},\mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))w$ involves only finitely many negative powers of x_i and y_j for $j \neq i$.

Lemma 2.16 Let $f(\mathbf{x}, \mathbf{y}) \in R(W, \mathbf{x}, \mathbf{y})$ be such that $(\mathbf{x} - \mathbf{y})^{\mathbf{m}} f(\mathbf{x}, \mathbf{y})$ exists for $\mathbf{m} \in \mathbf{Z}^n$. Then

$$\iota_{\alpha,\mathbf{x},\mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}+\mathbf{s}} f(\mathbf{x}, \mathbf{y})) = \iota_{\alpha,\mathbf{x},\mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}}) \iota_{\alpha,\mathbf{x},\mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{s}} f(\mathbf{x}, \mathbf{y})) \quad (2.54)$$

for $\mathbf{r}, \mathbf{s} \in \mathbf{Z}^n$.

Proof. In view of Lemma 2.14, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} (\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}}) &= (\mathbf{x} - \mathbf{y})^{k+\mathbf{r}}, \\ (\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{s}} f(\mathbf{x}, \mathbf{y})) &= (\mathbf{x} - \mathbf{y})^{k+\mathbf{s}} f(\mathbf{x}, \mathbf{y}), \\ (\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}+\mathbf{s}} f(\mathbf{x}, \mathbf{y})) &= (\mathbf{x} - \mathbf{y})^{\mathbf{r}+\mathbf{s}+k} f(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (2.55)$$

Then

$$\begin{aligned} &(\mathbf{x} - \mathbf{y})^{2k} \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}}) \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{s}} f(\mathbf{x}, \mathbf{y})) \\ &= (\mathbf{x} - \mathbf{y})^{k+\mathbf{r}} (\mathbf{x} - \mathbf{y})^{k+\mathbf{s}} f(\mathbf{x}, \mathbf{y}) \\ &= (\mathbf{x} - \mathbf{y})^{2k} (\mathbf{x} - \mathbf{y})^{\mathbf{r}+\mathbf{s}} f(\mathbf{x}, \mathbf{y}) \\ &= (\mathbf{x} - \mathbf{y})^{2k} \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{r}+\mathbf{s}} f(\mathbf{x}, \mathbf{y})). \end{aligned} \quad (2.56)$$

Now it follows immediately from Remark 2.1. \square

Remark 2.17 Let W and W' be given as in Remark 2.9. Let ψ, ϕ be mutually local homogeneous G_n -vertex operators. Let $w' \in W'$, $w \in W$ and let $h(\mathbf{x}, \mathbf{y})$ be as in Remark 2.9. Then for each $\alpha \in (F_2)^n$, $\langle w', \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))w \rangle$ are the formal Laurent series of the same rational function $h(\mathbf{x}, \mathbf{y})$ in a different domain.

Remark 2.18 We here give more information about $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))$. Recall that for $w \in W$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))w = \iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k}) \left((\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y})w \right), \quad (2.57)$$

where k is a nonnegative integer such that

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y}))). \quad (2.58)$$

With $(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y})w \in W((\mathbf{x}, \mathbf{y}))$, from (2.57) we see that each coefficient of formal series $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))w$ is a finite sum of the coefficients of $\psi(\mathbf{x})\phi(\mathbf{y})w$. Then the coefficients of $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))$ lie in the closure of the space spanned by the coefficients of $\psi(\mathbf{x})\phi(\mathbf{y})$ under a certain natural topology on $\text{End } W$.

Lemma 2.19 Let $f(\mathbf{x}, \mathbf{y}) \in R(W, \mathbf{x}, \mathbf{y})$. Then there exists $k \in \mathbb{N}$ such that

$$(x_i - y_i)^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y})) = (x_i - y_i)^k \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y})) \quad (2.59)$$

for $\alpha \in (F_2)^n$, $1 \leq i \leq n$. In particular, we may take $k \geq 0$ such that $(\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y})))$.

Proof. Let $k \geq 0$ be such that $(\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y})))$. By Lemma 2.14, we have

$$(\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y})) = (\mathbf{x} - \mathbf{y})^k f(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^k \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y})). \quad (2.60)$$

Set

$$p_i(\mathbf{x}, \mathbf{y}) = \prod_{j=1, j \neq i}^n (x_j - y_j)^{-k}. \quad (2.61)$$

It is easy to see that the associativity assumption holds for both the products

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(p_i(\mathbf{x}, \mathbf{y}))(\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y}))$$

and

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(p_i(\mathbf{x}, \mathbf{y}))(\mathbf{x} - \mathbf{y})^k \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}}(f(\mathbf{x}, \mathbf{y})).$$

Multiplying both sides of (2.60) by $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(p_i(\mathbf{x}, \mathbf{y}))$ and then applying associativity we obtain the desired identity. \square

Furthermore, we have:

Corollary 2.20 *Let ψ and ϕ be mutually local G_n -vertex operators on W . Then there exists $k \in \mathbb{N}$ such that*

$$(\mathbf{x} - \mathbf{y})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = (\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y}), \quad (2.62)$$

$$(x_i - y_i)^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = (x_i - y_i)^k \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) \quad (2.63)$$

for $\alpha \in (F_2)^n$, $1 \leq i \leq n$. In particular, we may take $k \geq 0$ such that (2.40) holds.

Proof. Let $k \in \mathbb{N}$ be such that (2.40) holds. Then

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x})\phi(\mathbf{y}) \in \text{Hom}(W, W((\mathbf{x}, \mathbf{y}))).$$

Now it follows immediately from Lemma 2.14 and Lemma 2.19. \square

Lemma 2.21 *Let ψ , ϕ and θ be pair-wise mutually local G_n -vertex operators on W . Then there exists $k \in \mathbb{N}$ such that*

$$(\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \theta(\mathbf{z}) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = (\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) \theta(\mathbf{z}) \quad (2.64)$$

for all $\alpha \in (F_2)^n$.

Proof. By Corollary 2.20 and the locality assumption, there exists $k \in \mathbb{N}$ such that (2.62) and the locality relation (2.40) for (θ, ψ) and (θ, ϕ) hold. Then

$$\begin{aligned} & (\mathbf{x} - \mathbf{y})^k (\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \theta(\mathbf{z}) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) \\ = & (\mathbf{x} - \mathbf{y})^k (\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \theta(\mathbf{z}) \psi(\mathbf{x})\phi(\mathbf{y}) \\ = & (\mathbf{x} - \mathbf{y})^k (\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \psi(\mathbf{x})\phi(\mathbf{y})\theta(\mathbf{z}) \\ = & (\mathbf{x} - \mathbf{y})^k (\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))\theta(\mathbf{z}). \end{aligned} \quad (2.65)$$

Notice that the associativity assumption holds for both the products

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k})(\mathbf{x} - \mathbf{y})^k \left((\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \theta(\mathbf{z}) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) \right)$$

and

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{-k})(\mathbf{x} - \mathbf{y})^k \left((\mathbf{x} - \mathbf{z})^k (\mathbf{y} - \mathbf{z})^k \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y}))\theta(\mathbf{z}) \right).$$

By taking $U = (\text{End } W)[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$ in Remark 2.1, we easily see that we can cancel the factor $(\mathbf{x} - \mathbf{y})^k$ from (2.65) to get the desired identity. \square

3 Higher dimensional analogues of vertex algebras

In this section we define a higher dimensional analogue of the notion of vertex algebra with a Jacobi identity as the main axiom and we prove basic duality properties. We also give some examples.

Definition 3.1 A G_n -vertex algebra is a vector space V equipped with a linear map

$$\begin{aligned} Y : \quad V &\rightarrow (\text{End } V)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]] \\ v &\mapsto Y(v, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} v_{\mathbf{m}} \mathbf{x}^{-\mathbf{m}-1} \quad (\text{where } v_{\mathbf{m}} \in \text{End } V) \end{aligned} \quad (3.1)$$

and equipped with a distinguished vector $\mathbf{1} \in V$ such that the following conditions hold:

- (A1) For $u \in V$, $Y(u, \mathbf{x}) \in VO_{G_n}(V)$, i.e., $Y(u, \mathbf{x})v \in V((\mathbf{x}))$ for $u, v \in V$.
- (A2) $Y(\mathbf{1}, \mathbf{x}) = 1$ and $Y(v, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]]$ with $\lim_{\mathbf{x} \rightarrow 0} Y(v, \mathbf{x})\mathbf{1} = v$ for $v \in V$.
- (A3) For $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x})Y(v, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y})Y(u, \mathbf{x}) \quad (3.2)$$

(the *weak commutativity*).

- (A4) For $u, v \in V$,

$$\begin{aligned} & \mathbf{y}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}} \right) Y(Y(u, \mathbf{z})v, \mathbf{y}) \\ = & \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x})Y(v, \mathbf{y}) \right) \end{aligned} \quad (3.3)$$

(the *Jacobi identity*).

Note that Axioms (A1) and (A3) are necessary for the Jacobi identity (Axiom (A4)) to make sense and that $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$ extends naturally on $R(V, \mathbf{x}, \mathbf{y})[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$.

Definition 3.2 Let V be a G_n -vertex algebra. For $1 \leq i \leq n$, we define $D_i \in \text{End } V$ by

$$D_i(v) = \text{Res}_{\mathbf{x}} x_i^{-1} \mathbf{x}^{-1} Y(v, \mathbf{x}) \mathbf{1} \quad \text{for } v \in V. \quad (3.4)$$

Then from (A2) we have

$$Y(v, \mathbf{y}) \mathbf{1} \equiv v + y_1 D_1(v) + \dots + y_n D_n(v) \quad \text{mod } \left(\sum_{i=1}^n y_i^2 V[[\mathbf{y}]] \right). \quad (3.5)$$

Set

$$\mathbf{x}\mathbf{D} = x_1 D_1 + \dots + x_n D_n. \quad (3.6)$$

Proposition 3.3 Let V be a vertex algebra and let D_i be defined as above. Then

$$[D_i, Y(u, \mathbf{x})] = Y(D_i(u), \mathbf{x}) = \partial_{x_i} Y(u, \mathbf{x}) \quad (3.7)$$

for $u \in V$.

Proof. Using the Jacobi identity with $v = \mathbf{1}$, (A2) and (2.32) we get

$$\begin{aligned} & Y(D_i(u), \mathbf{y}) \\ &= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} Y(Y(u, \mathbf{z}) \mathbf{1}, \mathbf{y}) \\ &= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x}) \right) \\ &= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} \left(\sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \right) \right) Y(u, \mathbf{x}) \\ &= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} \mathbf{y}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}} \right) Y(u, \mathbf{x}) \\ &= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} \mathbf{x}^{-1} \delta \left(\frac{\mathbf{y} + \mathbf{z}}{\mathbf{x}} \right) Y(u, \mathbf{x}) \\ &= \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} Y(u, \mathbf{y} + \mathbf{z}) \\ &= \text{Res}_{\mathbf{z}} z_i^{-1} \mathbf{z}^{-1} e^{(z_1 \partial_{y_1} + \dots + z_n \partial_{y_n})} Y(u, \mathbf{y}) \\ &= \partial_{y_i} Y(u, \mathbf{y}). \end{aligned} \quad (3.8)$$

This proves the second equality of (3.7).

Applying the Jacobi identity to $\mathbf{1}$ and then taking $\text{Res}_{\mathbf{x}}$ and using (3.5) we get

$$\begin{aligned}
& D_i(Y(u, \mathbf{z})v) \\
&= \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} Y(Y(u, \mathbf{z})v, \mathbf{y}) \mathbf{1} \\
&= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \right) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (Y(u, \mathbf{x})Y(v, \mathbf{y}) \mathbf{1}) \\
&= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \right) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (Y(v, \mathbf{y})Y(u, \mathbf{x}) \mathbf{1}) \\
&= \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} \mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x})Y(v, \mathbf{y}) \mathbf{1} \\
&= \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} Y(u, \mathbf{z} + \mathbf{y})Y(v, \mathbf{y}) \mathbf{1} \\
&= \text{Res}_{\mathbf{y}} y_i^{-1} \mathbf{y}^{-1} e^{(y_1 \partial_{z_1} + \dots + y_n \partial_{z_n})} Y(u, \mathbf{z})Y(v, \mathbf{y}) \mathbf{1} \\
&= Y(u, \mathbf{z})D_i(v) + \partial_{z_i} Y(u, \mathbf{z})v,
\end{aligned} \tag{3.9}$$

noting that because for $m \in \mathbf{Z}$, $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x} - \mathbf{y})^m$ involves only nonnegative powers of x_i for i with $\alpha_i = 1$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(Y(v, \mathbf{y})Y(u, \mathbf{x}) \mathbf{1})$$

involves only nonnegative powers of x_i for i with $\alpha_i = 1$,

$$\begin{aligned}
& \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m Y(u, \mathbf{x})Y(v, \mathbf{y}) \mathbf{1}) \\
&= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (Y(u, \mathbf{x})Y(v, \mathbf{y}) \mathbf{1}) \\
&= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (Y(v, \mathbf{y})Y(u, \mathbf{x}) \mathbf{1}) \\
&= 0
\end{aligned} \tag{3.10}$$

for all $m \in \mathbf{Z}$, $\alpha \neq (0, \dots, 0)$. Then

$$D_i Y(u, \mathbf{z})v - Y(u, \mathbf{z})D_i(v) = \partial_{z_i} Y(u, \mathbf{z})v. \tag{3.11}$$

This proves the outside equality of (3.7). Now the proof is complete. \square

From (3.7) and the Taylor Theorem we have the following conjugation formula:

$$e^{\mathbf{y}\mathbf{D}} Y(u, \mathbf{x}) e^{-\mathbf{y}\mathbf{D}} = Y(u, \mathbf{x} + \mathbf{y}) \quad \text{for } u \in V. \tag{3.12}$$

We also have the following skew-symmetry:

Proposition 3.4 *In the setting of Proposition 3.3 we have*

$$Y(u, \mathbf{x})v = e^{\mathbf{x}\mathbf{D}} Y(v, -\mathbf{x})u \quad \text{for } u, v \in V. \tag{3.13}$$

Proof. Define a translation σ of $(F_2)^n$ by

$$\sigma(\alpha) = \alpha + (1, \dots, 1) \quad \text{for } \alpha \in (F_2)^n. \quad (3.14)$$

Then $(-1)^{|\sigma(\alpha)|} = (-1)^n(-1)^{|\alpha|}$ and

$$\iota_{\alpha, \mathbf{y}, \mathbf{x}} = \iota_{\sigma(\alpha), \mathbf{x}, \mathbf{y}} \quad \text{for } \alpha \in (F_2)^n. \quad (3.15)$$

It follows that the right-hand side of the Jacobi identity stays the same if we replace $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ by $(\mathbf{y}, \mathbf{x}, -\mathbf{z})$. Just like in the ordinary case [FHL], using this, the Jacobi identity and the creation property we obtain the skew-symmetry (3.13). \square

Similar to the ordinary case, we have the following weak associativity:

Proposition 3.5 *Let V be a vertex algebra. For $u, w \in V$, there exists $l \geq 0$ such that*

$$(\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w = (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z})v, \mathbf{y}) w \quad (3.16)$$

for all $v \in V$. In particular, we may take $l \geq 0$ such that $\mathbf{x}^l Y(u, \mathbf{x})w \in V[[\mathbf{x}]]$. On the other hand, in Definition 3.1 let us only assume (A1)-(A3), in addition we assume the weak associativity property. Then the Jacobi identity holds, so that V is a vertex algebra.

Proof. Assume the Jacobi identity. For any $l \in \mathbb{N}$, applying $\text{Res}_{\mathbf{x}} \mathbf{x}^l$ to both sides of the Jacobi identity and using delta function substitution property we get

$$\begin{aligned} & (\mathbf{y} + \mathbf{z})^l Y(Y(u, \mathbf{z})v, \mathbf{y}) \\ &= \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \mathbf{x}^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w \right). \end{aligned} \quad (3.17)$$

Let $l \in \mathbb{N}$ be such that $\mathbf{x}^l Y(u, \mathbf{x})w \in V[[\mathbf{x}]]$. Recall (2.50):

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(Y(u, \mathbf{x}) Y(v, \mathbf{y})) = \iota_{\alpha, \mathbf{x}, \mathbf{y}}(Y(v, \mathbf{y}) Y(u, \mathbf{x})).$$

Because for $m \in \mathbf{Z}$, $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x} - \mathbf{y})^m$ involves only nonnegative powers of x_i for i with $\alpha_i = 1$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x}^l Y(v, \mathbf{y}) Y(u, \mathbf{x}) w)$$

involves only nonnegative powers of x_i for i with $\alpha_i = 1$. Then

$$\begin{aligned} & \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m \mathbf{x}^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w) \\ &= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (\mathbf{x}^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w) \\ &= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}} (\mathbf{x}^l Y(v, \mathbf{y}) Y(u, \mathbf{x}) w) \\ &= 0 \end{aligned} \quad (3.18)$$

for all $m \in \mathbf{Z}$, $\alpha \neq (0, \dots, 0)$. Therefore

$$\begin{aligned}
& (\mathbf{y} + \mathbf{z})^l (Y(Y(u, \mathbf{z})v, \mathbf{y})w \\
= & \text{Res}_{\mathbf{x}} \mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \mathbf{x}^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w \\
= & \text{Res}_{\mathbf{x}} \mathbf{x}^{-1} \delta \left(\frac{\mathbf{z} + \mathbf{y}}{\mathbf{x}} \right) (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w \\
= & (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w.
\end{aligned} \tag{3.19}$$

Now assume (3.16). Set

$$p(\mathbf{x}, \mathbf{y}) = \mathbf{x}^l (\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) w \in W((\mathbf{x}, \mathbf{y})) \tag{3.20}$$

and

$$h(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^{-l} \mathbf{z}^{-k} p(\mathbf{x}, \mathbf{y}) \in W((\mathbf{x}, \mathbf{y}, \mathbf{z})). \tag{3.21}$$

Then

$$Y(u, \mathbf{x}) Y(v, \mathbf{y}) w = h(\mathbf{x}, \mathbf{y}, \mathbf{x} - \mathbf{y}), \tag{3.22}$$

$$Y(Y(u, \mathbf{z})v, \mathbf{y}) w = h(\mathbf{y} + \mathbf{z}, \mathbf{y}, z). \tag{3.23}$$

Then the Jacobi identity immediately follows from (2.36). \square

Taking $\text{Res}_{\mathbf{z}}$ of the Jacobi identity we get

$$\sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} (Y(u, \mathbf{x}) Y(v, \mathbf{y})) = \text{Res}_{\mathbf{z}} \mathbf{y}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{z}}{\mathbf{y}} \right) Y(Y(u, \mathbf{z})v, \mathbf{y}). \tag{3.24}$$

This is a generalization of Borcherds' commutator formula. By taking $\text{Res}_{\mathbf{x}}$ we also have the following iterate formula:

$$Y(Y(u, \mathbf{z})v, \mathbf{y}) = \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x}) Y(v, \mathbf{y}) \right). \tag{3.25}$$

Note that the above iterate formula gives rise to an associator formula:

$$\begin{aligned}
& Y(Y(u, \mathbf{z})v, \mathbf{y}) - Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) \\
= & \sum_{\alpha \in (F_2)^n, \alpha \neq 0} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x}) Y(v, \mathbf{y}) \right).
\end{aligned} \tag{3.26}$$

But (3.24) does not give a genuine commutator formula. Nevertheless, similar to the ordinary vertex algebra case, we have:

Proposition 3.6 *Let $u, v \in V$. Then $[Y(u, \mathbf{x}), Y(v, \mathbf{y})] = 0$ if and only if $Y(u, \mathbf{x})v \in V[[\mathbf{x}]]$.*

Proof. Suppose that $[Y(u, \mathbf{x}), Y(v, \mathbf{y})] = 0$. Then with $Y(u, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]]$, we have

$$Y(u, \mathbf{x})v = \text{Res}_{\mathbf{y}} \mathbf{y}^{-1} Y(u, \mathbf{x})Y(v, \mathbf{y})\mathbf{1} = \text{Res}_{\mathbf{y}} \mathbf{y}^{-1} Y(v, \mathbf{y})Y(u, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]].$$

Conversely, assume $Y(u, \mathbf{x})v \in V[[\mathbf{x}]]$. In view of Proposition 3.5, for any $w \in V$, we have

$$Y(u, \mathbf{z} + \mathbf{y})Y(w, \mathbf{y})v = Y(Y(u, \mathbf{z})w, \mathbf{y})v. \quad (3.27)$$

Applying $e^{-\mathbf{y}\mathbf{D}}$ from left to (3.27) and then using the conjugation formula (3.12) and the skew-symmetry (3.13) (twice) we obtain

$$Y(u, \mathbf{z})Y(v, -\mathbf{y})w = Y(v, -\mathbf{y})Y(u, \mathbf{z})w. \quad (3.28)$$

This proves that $[Y(u, \mathbf{x}), Y(v, \mathbf{y})] = 0$. Now the proof is complete. \square

Similar to the ordinary case ([FHL], [DL], [Li2], cf. [Ka]) (in fact, using exactly the same argument of [Li2]) we have:

Theorem 3.7 *Let $(V, Y, \mathbf{1}, \mathbf{D})$ be a quadruple consisting of a vector space V , a vector $\mathbf{1}$ in V , a linear map Y from V to $(\text{End } V)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$ and a vector*

$$\mathbf{D} = (D_1, \dots, D_n) \in (\text{End } V)^n$$

such that $D_i(\mathbf{1}) = 0$ for $i = 1, \dots, n$ and such that the following conditions hold:

- (E1) $Y(u, \mathbf{x})v \in V((\mathbf{x}))$ for $u, v \in V$.
- (E2) $Y(\mathbf{1}, \mathbf{x}) = \mathbf{1}$ and $Y(v, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]]$ with $\lim_{\mathbf{x} \rightarrow 0} Y(v, \mathbf{x})\mathbf{1} = v$ for $v \in V$.
- (E3) $[D_i, Y(u, \mathbf{x})] = \partial_{x_i} Y(u, \mathbf{x})$.
- (E4) For $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x})Y(v, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y})Y(u, \mathbf{x}).$$

Then $(V, Y, \mathbf{1})$ is a vertex algebra and $D_i(v) = \text{Res}_{\mathbf{x}} x_i^{-1} \mathbf{x}^{-1} Y(v, \mathbf{x})\mathbf{1}$ for $v \in V$.

Proof. First, from (E3) and the Taylor Theorem we immediately have (cf. (3.12))

$$e^{\mathbf{y}\mathbf{D}} Y(v, \mathbf{x}) e^{-\mathbf{y}\mathbf{D}} = Y(v, \mathbf{x} + \mathbf{y}) \quad \text{for } v \in V. \quad (3.29)$$

Applying (3.29) to $\mathbf{1}$, using the fact that $D_i(\mathbf{1}) = 0$ for $i = 1, \dots, n$, we get

$$e^{\mathbf{y}\mathbf{D}} Y(v, \mathbf{x})\mathbf{1} = Y(v, \mathbf{x} + \mathbf{y})\mathbf{1}. \quad (3.30)$$

Since $Y(v, \mathbf{x} + \mathbf{y})\mathbf{1} \in V[[\mathbf{x} + \mathbf{y}]]$, we may set $\mathbf{y} = -\mathbf{x}$. Then using the creation property we get $e^{-\mathbf{x}\mathbf{D}} Y(v, \mathbf{x})\mathbf{1} = v$. That is,

$$Y(v, \mathbf{x})\mathbf{1} = e^{\mathbf{x}\mathbf{D}} v. \quad (3.31)$$

Second, we have the skew-symmetry as follows: Let $k \in \mathbb{N}$ be such that the weak commutativity holds and such that $\mathbf{x}^k Y(v, \mathbf{x}) u \in V[[\mathbf{x}]]$. Then

$$\begin{aligned} (\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) \mathbf{1} &= (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) \mathbf{1} \\ &= (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) e^{\mathbf{x}\mathbf{D}} u \\ &= (\mathbf{x} - \mathbf{y})^k e^{\mathbf{x}\mathbf{D}} Y(v, \mathbf{y} - \mathbf{x}) u. \end{aligned} \quad (3.32)$$

Since $(\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y} - \mathbf{x}) u$ involves only nonnegative powers of $(\mathbf{x} - \mathbf{y})$, we may set \mathbf{y} to zero. Setting $\mathbf{y} = 0$ and using creation property we get

$$\mathbf{x}^k Y(u, \mathbf{x}) v = \mathbf{x}^k e^{\mathbf{x}\mathbf{D}} Y(v, -\mathbf{x}) u, \quad (3.33)$$

which immediately gives (cf. (3.13))

$$Y(u, \mathbf{x}) v = e^{\mathbf{x}\mathbf{D}} Y(v, -\mathbf{x}) u \quad \text{for } u, v \in V. \quad (3.34)$$

Third, let $l \in \mathbb{N}$ be such that the weak commutativity holds for (u, w) . Then using the skew-symmetry and conjugation formula we obtain the weak associativity as

$$\begin{aligned} (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w &= (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) e^{-\mathbf{y}\mathbf{D}} Y(w, -\mathbf{y}) v \\ &= (\mathbf{z} + \mathbf{y})^l e^{-\mathbf{y}\mathbf{D}} Y(u, \mathbf{z}) Y(w, -\mathbf{y}) v \\ &= (\mathbf{z} + \mathbf{y})^l e^{-\mathbf{y}\mathbf{D}} Y(w, -\mathbf{y}) Y(u, \mathbf{z}) v \\ &= (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z}) v, \mathbf{y}) w. \end{aligned} \quad (3.35)$$

Now it follows from the second part of Proposition 3.5 that $(V, Y, \mathbf{1})$ is a vertex algebra. The last assertion follows from (E3) and the assumption that $D_i(\mathbf{1}) = 0$ for $i = 1, \dots, n$. \square

Remark 3.8 As proved in [DL], [FHL] and [Li2] for the ordinary case, in view of Theorem 3.7, one may define the notion of vertex algebra by using Conditions (E1)-(E4) as the axioms (see for example [Ka] in the ordinary case).

Remark 3.9 In fact, in Theorem 3.7, (E1) follows from (E2) and (E4), so that (E1) is redundant. Indeed, for $u, v \in V$, let $k \in \mathbb{N}$ be such that the weak commutativity relation holds for (u, v) . Then

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) \mathbf{1} = (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) \mathbf{1}. \quad (3.36)$$

Since the expression on the left-hand side involves only nonnegative powers of y_i 's and the expression on the right-hand side involves only nonnegative powers of x_i 's, we have

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) \mathbf{1} \in V[[\mathbf{x}, \mathbf{y}]]. \quad (3.37)$$

Setting $\mathbf{y} = 0$, then using creation property, we get

$$\mathbf{x}^k Y(u, \mathbf{x})v \in V[[\mathbf{x}]]. \quad (3.38)$$

That is, $Y(u, \mathbf{x})v \in V((\mathbf{x}))$.

Proposition 3.10 *In the definition of the notion G_n -vertex algebra, let us only assume (A1)-(A2) and in addition we assume that the skew-symmetry (3.13) and weak associativity hold. Then weak commutativity and the Jacobi identity hold.*

Proof. Let $u, v, w \in W$ and let $l \in \mathbb{N}$ be such that

$$(\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w = (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z})v, y) w. \quad (3.39)$$

Applying $e^{-\mathbf{y}\mathbf{D}}$ from left and then using skew-symmetry (3.13) we get

$$\begin{aligned} & (\mathbf{z} + \mathbf{y})^l e^{-\mathbf{y}\mathbf{D}} Y(u, \mathbf{z} + \mathbf{y}) e^{\mathbf{y}\mathbf{D}} Y(w, -\mathbf{y}) v \\ &= (\mathbf{z} + \mathbf{y})^l Y(w, -\mathbf{y}) Y(u, \mathbf{z}) v. \end{aligned} \quad (3.40)$$

In particular, for $w = \mathbf{1}$, we obtain

$$(\mathbf{z} + \mathbf{y})^l e^{-\mathbf{y}\mathbf{D}} Y(u, \mathbf{z} + \mathbf{y}) e^{\mathbf{y}\mathbf{D}} v = (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z}) v, \quad (3.41)$$

which immediately implies the conjugation formula (3.12).

Now, we go back to let w be arbitrary again. From (3.40) using the conjugation formula (3.12) we get

$$(\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z}) Y(w, -\mathbf{y}) v = (\mathbf{z} + \mathbf{y})^l Y(w, -\mathbf{y}) Y(u, \mathbf{z}) v. \quad (3.42)$$

Notice that l depends only on u and w , not v . This gives weak commutativity. The rest of the assertions immediately follows. \square

Remark 3.11 Any G_n -vertex algebra V is a natural G_{n+1} -vertex algebra. Indeed, Axioms (A1)-(A3) clearly hold and for $u, v \in V$. Furthermore, multiplying both sides of (3.3) by $y_{n+1}^{-1} \delta\left(\frac{x_{n+1}-z_{n+1}}{y_{n+1}}\right)$ and then using the identity

$$y_{n+1}^{-1} \delta\left(\frac{x_{n+1}-z_{n+1}}{y_{n+1}}\right) = z_{n+1}^{-1} \delta\left(\frac{x_{n+1}-y_{n+1}}{z_{n+1}}\right) - z_{n+1}^{-1} \delta\left(\frac{-y_{n+1}+x_{n+1}}{z_{n+1}}\right) \quad (3.43)$$

to rewrite the new right-hand side we obtain the Jacobi identity for G_{n+1} -vertex algebras.

On the other hand, let $(V, Y, \mathbf{1})$ be a G_n -vertex algebra. For any subset I of $\{1, \dots, n\}$ we set

$$V_I = \{v \in V \mid D_i(v) = 0 \text{ for } i \in I\}. \quad (3.44)$$

In view of Proposition 3.3, for $v \in V$, $v \in V_I$ if and only if

$$\partial_{x_i} Y(v, \mathbf{x}) = 0 \quad \text{for } i \in I. \quad (3.45)$$

Notice that for $u, v \in V_I$,

$$\left(\coprod_{i \in I} \text{Res}_{x_i} \right) \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) Y(u, \mathbf{x}) Y(v, \mathbf{y}) \right) = 0 \quad (3.46)$$

for $\alpha \in (F_2)^n$ with $\alpha_i = 1$ for some $i \in I$. Then by taking $\coprod_{i \in I} \text{Res}_{x_i}$ of (3.3) we obtain the Jacobi identity for $G_{n-|I|}$ -vertex algebras. Then it follows that $(V_I, Y, \mathbf{1})$ is a $G_{n-|I|}$ -vertex algebra.

Definition 3.12 Let V be a vertex algebra. A V -module is a vector space W equipped with a linear map Y from V to $(\text{End } W)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$ that satisfies the following conditions:

- (M1) $Y(v, \mathbf{x})w \in W((\mathbf{x}))$ for $v \in V$, $w \in W$.
- (M2) $Y(\mathbf{1}, \mathbf{x}) = 1$.
- (M3) The weak commutativity (A3) on W holds.
- (M4) The Jacobi identity (3.3) on W holds.

Recall that Jacobi identity implies weak associativity. In fact, in the notion of module, weak associativity implies weak commutativity and Jacobi identity (cf. [Le], [Li2]).

Proposition 3.13 Let V be a vertex algebra and let W be a vector space equipped with a linear map Y from V to $(\text{End } W)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$ such that (M1) and (M2) of Definition 3.12 hold and such that the following weak associativity holds: For every $u \in V$, $w \in W$, there exists $l \in \mathbb{N}$ such that

$$(\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w = (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z})v, y) w \quad (3.47)$$

for all $v \in V$. Then W is a V -module.

Proof. In view of Proposition 3.5 (the second part), it suffices to prove the weak commutativity.

Let $u, v \in V$, $w \in W$. From (M1) and the given weak associativity, there exists $k \in \mathbb{N}$ (only depending on u, v) such that

$$\mathbf{z}^k Y(u, \mathbf{z})v \in V[[\mathbf{z}]]$$

and there exists $l \in \mathbb{N}$ such that all the following conditions hold:

$$\mathbf{y}^l Y(v, \mathbf{y})w \in W[[\mathbf{y}]], \quad (3.48)$$

$$(\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w = (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z})v, \mathbf{y}) w, \quad (3.49)$$

$$(\mathbf{x} - \mathbf{z})^l Y(v, -\mathbf{z} + \mathbf{x}) Y(u, \mathbf{x}) w = (\mathbf{x} - \mathbf{z})^l Y(Y(v, -\mathbf{z})u, \mathbf{x}) w. \quad (3.50)$$

Then

$$\mathbf{y}^l \mathbf{z}^k (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w = \mathbf{y}^l \mathbf{z}^k (\mathbf{z} + \mathbf{y})^l Y(Y(u, \mathbf{z}) v, \mathbf{y}) w \in W[[\mathbf{y}, \mathbf{z}]].$$

Set

$$p(\mathbf{y}, \mathbf{z}) = \mathbf{y}^l \mathbf{z}^k (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w. \quad (3.51)$$

Then

$$\mathbf{y}^l (\mathbf{x} - \mathbf{y})^k \mathbf{x}^l Y(u, \mathbf{x}) Y(v, \mathbf{y}) w = p(\mathbf{y}, \mathbf{x} - \mathbf{y}) \quad (3.52)$$

because

$$\begin{aligned} & \mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) w \\ &= \mathbf{x}^{-1} \delta \left(\frac{\mathbf{z} + \mathbf{y}}{\mathbf{x}} \right) \mathbf{y}^l \mathbf{z}^k (\mathbf{z} + \mathbf{y})^l Y(u, \mathbf{z} + \mathbf{y}) Y(v, \mathbf{y}) w \\ &= \mathbf{x}^{-1} \delta \left(\frac{\mathbf{z} + \mathbf{y}}{\mathbf{x}} \right) p(\mathbf{y}, \mathbf{z}) \\ &= \mathbf{x}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) p(\mathbf{y}, \mathbf{x} - \mathbf{y}). \end{aligned} \quad (3.53)$$

On the other hand, using skew-symmetry and Taylor theorem we obtain

$$\begin{aligned} & \mathbf{z}^{-1} \delta \left(\frac{\mathbf{y} - \mathbf{x}}{-\mathbf{z}} \right) \mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) w \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) \mathbf{x}^l (-\mathbf{z} + \mathbf{x})^l \mathbf{z}^k Y(v, -\mathbf{z} + \mathbf{x}) Y(u, \mathbf{x}) w \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) \left(\mathbf{x}^l (-\mathbf{z} + \mathbf{x})^l \mathbf{z}^k Y(Y(v, -\mathbf{z}) u, \mathbf{x}) w \right) \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) \left(\mathbf{x}^l \mathbf{z}^k (-\mathbf{z} + \mathbf{x})^l Y(e^{-\mathbf{z}\mathbf{D}} Y(u, \mathbf{z}) v, \mathbf{x}) w \right) \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) \left(\mathbf{x}^l \mathbf{z}^k (-\mathbf{z} + \mathbf{x})^l Y(Y(u, \mathbf{z}) v, \mathbf{x} - \mathbf{z}) w \right) \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) \left(\mathbf{y}^l \mathbf{z}^l (\mathbf{y} + \mathbf{z})^l Y(Y(u, \mathbf{z}) v, \mathbf{y}) w \right) |_{\mathbf{y}=\mathbf{x}-\mathbf{z}} \\ &= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) p(\mathbf{y}, \mathbf{z}) |_{\mathbf{y}=\mathbf{x}-\mathbf{z}}. \end{aligned} \quad (3.54)$$

Since $p(\mathbf{y}, \mathbf{z}) \in W[[\mathbf{y}, \mathbf{z}]]$,

$$p(\mathbf{y}, \mathbf{z}) |_{\mathbf{y}=\mathbf{x}-\mathbf{z}} = p(\mathbf{y}, \mathbf{z}) |_{\mathbf{y}=-\mathbf{z}+\mathbf{x}}.$$

Then

$$\begin{aligned}
& \mathbf{z}^{-1} \delta \left(\frac{\mathbf{y} - \mathbf{x}}{-\mathbf{z}} \right) \mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^l Y(v, \mathbf{y}) Y(u, \mathbf{x}) w \\
&= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) p(\mathbf{y}, \mathbf{z})|_{\mathbf{y} = -\mathbf{z} + \mathbf{x}} \\
&= \mathbf{y}^{-1} \delta \left(\frac{-\mathbf{z} + \mathbf{x}}{\mathbf{y}} \right) p(\mathbf{y}, \mathbf{z}) \\
&= \mathbf{z}^{-1} \delta \left(\frac{\mathbf{y} - \mathbf{x}}{-\mathbf{z}} \right) p(\mathbf{y}, -\mathbf{y} + \mathbf{x}) \\
&= \mathbf{z}^{-1} \delta \left(\frac{\mathbf{y} - \mathbf{x}}{-\mathbf{z}} \right) p(\mathbf{y}, \mathbf{x} - \mathbf{y}). \tag{3.55}
\end{aligned}$$

Then

$$\mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) w = p(\mathbf{y}, -\mathbf{y} + \mathbf{x}). \tag{3.56}$$

Consequently,

$$\mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) w = \mathbf{x}^l \mathbf{y}^l (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) w. \tag{3.57}$$

Thus

$$(\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{x}) Y(v, \mathbf{y}) w = (\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{y}) Y(u, \mathbf{x}) w.$$

Recall that k does not depend on w . This proves the weak commutativity and concludes the proof. \square

Remark 3.14 Given a G_n -vertex algebra V , we have seen that in the notion of V -module, weak associativity is stronger than weak commutativity. This is contrary to the situation for the notion of G_n -vertex algebra where the vacuum vector plays a crucial role in the proof.

We later shall need the following result (cf. [Li1]):

Lemma 3.15 *Let V be a G_n -vertex algebra and S a generating set of V . Let W be a V -module and let $w \in W$ be such that $Y(u, \mathbf{x})w \in W[[\mathbf{x}]]$ for $u \in S$. Then $Y(v, \mathbf{x})w \in V[[\mathbf{x}]]$ for all $v \in V$. Furthermore, the linear map*

$$f : V \rightarrow W; v \mapsto \lim_{\mathbf{x} \rightarrow 0} Y(v, \mathbf{x}) w \tag{3.58}$$

is a V -homomorphism.

Proof. For the first part, it suffices to prove that all elements v of V satisfying $Y(v, \mathbf{x})w \in W[[\mathbf{x}]]$ form a vertex subalgebra of V . Furthermore, it is enough to prove that if $Y(u, \mathbf{x})w, Y(v, \mathbf{x})w \in W[[\mathbf{x}]]$, then $Y(u_{\mathbf{m}}v, \mathbf{x})w \in W[[\mathbf{x}]]$ for $\mathbf{m} \in \mathbf{Z}^n$. In view of Proposition 3.5 we have

$$Y(Y(u, \mathbf{z})v, \mathbf{y})w = Y(u, \mathbf{z} + \mathbf{y})Y(v, \mathbf{y})w \quad \text{for } u, v \in V. \quad (3.59)$$

Then $Y(u_{\mathbf{m}}v, \mathbf{x})w \in W[[\mathbf{x}]]$ for $\mathbf{m} \in \mathbf{Z}^n$.

By taking $\lim_{\mathbf{x} \rightarrow 0}$ in (3.59) we obtain

$$f(Y(u, \mathbf{z})v) = Y(u, \mathbf{z})f(v) \quad \text{for } u, v \in V. \quad (3.60)$$

That is, f is a V -homomorphism from V to W . \square

Furthermore we have:

Proposition 3.16 *Let V, S, W and w be given as in Lemma 3.15. Furthermore, assume that W is a faithful V -module in the sense that Y is one-to-one, w generates W as a V -module and assume that there exists $\mathbf{d} = (d_1, \dots, d_n) \in (\text{End } W)^n$ such that $d_i(w) = 0$ and*

$$[d_i, Y(v, \mathbf{x})] = \partial_{x_i} Y(v, \mathbf{x}) \quad (3.61)$$

on W for $i = 1, \dots, n$, $v \in V$. Then the linear map f defined in Lemma 3.15 is a V -isomorphism.

Proof. In view of Lemma 3.15, from the assumption that w generates W as a V -module, f is an onto V -homomorphism. It remains to prove that f is one-to-one. Assume that $f(v) = 0$ for some $v \in V$. From the assumptions we get

$$Y(v, \mathbf{x} + \mathbf{y})w = e^{\mathbf{y}\mathbf{D}}Y(v, \mathbf{x})e^{-\mathbf{y}\mathbf{D}}w = e^{\mathbf{y}\mathbf{D}}Y(v, \mathbf{x})w. \quad (3.62)$$

With $Y(v, \mathbf{x} + \mathbf{y})w \in W[[\mathbf{x} + \mathbf{y}]]$, we may set $\mathbf{x} = \mathbf{0}$ to obtain $Y(v, \mathbf{y})w = e^{\mathbf{y}\mathbf{D}}f(v) = 0$. Let $a \in W$ be such that $Y(v, \mathbf{x})a = 0$ and let $u \in S$. Then there exists $k \in \mathbb{N}$ such that

$$(\mathbf{x} - \mathbf{y})^k Y(v, \mathbf{x})Y(u, \mathbf{y})a = (\mathbf{x} - \mathbf{y})^k Y(u, \mathbf{y})Y(v, \mathbf{x})a = 0.$$

Thus $Y(v, \mathbf{x})Y(u, \mathbf{y})a = 0$. Then it follows from induction that $Y(v, \mathbf{x}) = 0$ on W . With W being faithful, we must have $v = 0$. This proves that f is one-to-one. \square

Note: In fact, in Proposition 3.16, it is enough to assume that (3.61) holds for $v \in S$. (See the proof Proposition 4.10.)

In the following we shall give some examples of G_n -vertex algebras.

Example 3.17 Let A be a commutative associative algebra with identity 1 equipped with n (possibly the same) pairwise commuting derivations D_1, \dots, D_n . Set

$$\mathbf{D} = (D_1, \dots, D_n). \quad (3.63)$$

Then as a convention we write

$$\mathbf{x}\mathbf{D} = x_1D_1 + \dots + x_nD_n. \quad (3.64)$$

For $a \in A$, we define

$$Y(a, \mathbf{x}) \in (\text{End } A)[[x_1, \dots, x_n]] \subset (\text{End } A)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$$

by

$$Y(a, \mathbf{x})b = (e^{\mathbf{x}\mathbf{D}}a)b = (e^{x_1D_1+\dots+x_nD_n}a)b \quad (3.65)$$

for $b \in A$. Then

$$Y(a, \mathbf{x})Y(b, \mathbf{y})c = (e^{x_1D_1+\dots+x_nD_n}a)(e^{y_1D_1+\dots+y_nD_n}b)c \quad (3.66)$$

for $a, b, c \in A$. Since D_1, \dots, D_n are pairwise commuting, we have

$$Y(a, \mathbf{x})Y(b, \mathbf{y}) = Y(b, \mathbf{y})Y(a, \mathbf{x}) \quad \text{for } a, b \in A. \quad (3.67)$$

In particular, the weak commutativity holds. Clearly, the vacuum properties hold with $\mathbf{1} = 1$. Then

$$\begin{aligned} \iota_{\alpha, \mathbf{x}, \mathbf{y}}(Y(a, \mathbf{x})Y(b, \mathbf{y})) &= Y(a, \mathbf{x})Y(b, \mathbf{y}) \\ &= (e^{x_1D_1+\dots+x_nD_n}a)(e^{y_1D_1+\dots+y_nD_n}b). \end{aligned} \quad (3.68)$$

On the other hand,

$$\begin{aligned} Y(Y(a, \mathbf{z})b, \mathbf{y})c &= (e^{y_1D_1+\dots+y_nD_n}(e^{z_1D_1+\dots+z_nD_n}a)b)c \\ &= (e^{(y_1+z_1)D_1+\dots+(y_n+z_n)D_n}a)(e^{y_1D_1+\dots+y_nD_n}b)c. \end{aligned} \quad (3.69)$$

Then

$$\begin{aligned} &\mathbf{y}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{z}}{\mathbf{y}}\right)Y(Y(a, \mathbf{z})b, \mathbf{y})c \\ &= \mathbf{y}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{z}}{\mathbf{y}}\right)(e^{(\mathbf{y}+\mathbf{z})\mathbf{D}}a)(e^{\mathbf{y}\mathbf{D}}b)c \\ &= \mathbf{x}^{-1}\delta\left(\frac{\mathbf{y}+\mathbf{z}}{\mathbf{x}}\right)(e^{\mathbf{x}\mathbf{D}}a)(e^{\mathbf{y}\mathbf{D}}b)c. \end{aligned} \quad (3.70)$$

Then the Jacobi identity immediately follows.

Remark 3.18 Let V be a G_n -vertex algebra such that

$$Y(v, \mathbf{x}) \in (\text{End } V)[[x_1, \dots, x_n]] \quad \text{for all } v \in V. \quad (3.71)$$

In view of Proposition 3.6 we have

$$[Y(u, \mathbf{x}), Y(v, \mathbf{y})] = 0 \quad \text{for } u, v \in V. \quad (3.72)$$

In view of Proposition 3.5 we have

$$Y(Y(u, \mathbf{z})v, \mathbf{y})w = Y(u, \mathbf{z} + \mathbf{y})Y(v, \mathbf{y})w \quad \text{for } u, v, w \in V. \quad (3.73)$$

Then one can easily see that V equipped with the product

$$u \cdot v = \text{Res}_{\mathbf{x}} \mathbf{x}^{-1} Y(u, \mathbf{x})v \quad \text{for } u, v \in V$$

is a commutative associative algebra with $\mathbf{1}$ as identity. Furthermore, D_i are mutually commuting derivations where

$$D_i(v) = \text{Res}_{\mathbf{x}} \mathbf{x}^{-1} x_i^{-1} Y(v, \mathbf{x})\mathbf{1} \quad \text{for } v \in V.$$

Then $Y(u, \mathbf{x})v = (e^{\mathbf{x}\mathbf{D}}u)v$ for $u, v \in V$.

Example 3.19 Let V_1, \dots, V_n be ordinary vertex algebras. Set

$$V = V_1 \otimes \cdots \otimes V_n \quad (3.74)$$

as a vector space. Define

$$Y(v, \mathbf{x}) = Y(v^1, x_1) \otimes \cdots \otimes Y(v^n, x_n) \in (\text{End } V)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]] \quad (3.75)$$

for $v = v^1 \otimes \cdots \otimes v^n \in V$. Use linearity to extend Y to be a linear map from V to $(\text{End } V)[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$. Set

$$\mathbf{1} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \in V. \quad (3.76)$$

Now we show that $(V, Y, \mathbf{1})$ is a G_n -vertex algebra. Clearly, the lower truncation condition and the vacuum property hold.

For $u = u^1 \otimes \cdots \otimes u^n$, $v = v^1 \otimes \cdots \otimes v^n \in V$, we have

$$Y(u, \mathbf{x})Y(v, \mathbf{y}) = Y(u^1, x_1)Y(v^1, y_1) \otimes \cdots \otimes Y(u^n, x_n)Y(v^n, y_n). \quad (3.77)$$

Then weak commutativity obviously holds. Furthermore, we have

$$\begin{aligned} & Y(Y((u^1 \otimes \cdots \otimes u^n), \mathbf{z})(v^1 \otimes \cdots \otimes v^n), \mathbf{y}) \\ &= Y(Y(u^1, z_1)v^1 \otimes \cdots \otimes Y(u^n, z_n)v^n, \mathbf{y}) \\ &= Y(Y(u^1, z_1)v^1, y_1) \otimes \cdots \otimes Y(Y(u^n, z_n)v^n, y_n). \end{aligned} \quad (3.78)$$

For two formal series $A(x), B(x) \in (\text{End } U)[[x, x^{-1}]]$, we define

$$\tau_0(A(x), B(y)) = A(x)B(y), \quad \tau_1(A(x), B(y)) = B(y)A(x). \quad (3.79)$$

Then for $\alpha = (\alpha_1, \dots, \alpha_n) \in (F_2)^n$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(Y(u, \mathbf{x})Y(v, \mathbf{y})) = \tau_{\alpha_1}(Y(u^1, x_1), Y(v^1, y_1)) \otimes \cdots \otimes \tau_{\alpha_n}(Y(u^n, x_n), Y(v^n, y_n)). \quad (3.80)$$

With this, we easily see that the Jacobi identity is exactly the tensor product of n ordinary Jacobi identities.

Furthermore, let W_i be a V_i -module for $i = 1, \dots, n$. Then $W_1 \otimes \cdots \otimes W_n$ is a V -module with the obvious action.

By considering certain subalgebras of the tensor product G_n -vertex algebras we can obtain G_n -vertex algebras a little bit more general than the tensor product G_n -vertex algebras.

Example 3.20 Let G be an abelian group and let $U = \bigoplus_{g \in G} U_g$ and $V = \bigoplus_{g \in G} V_g$ ordinary vertex algebras graded by G such that

$$u_{\mathbf{m}} u' \in U_{g+h} \quad \text{for } u \in U_g, u' \in U_h, g, h \in G, \quad (3.81)$$

$$v_{\mathbf{m}} v' \in V_{g+h} \quad \text{for } v \in V_g, v' \in V_h, g, h \in G. \quad (3.82)$$

Set

$$(U \otimes V)_G = \bigoplus_{g \in G} U_g \otimes V_g. \quad (3.83)$$

It is easy to see that $(U \otimes V)_G$ is a vertex subalgebra of $U \otimes V$.

A *homomorphism* from a G_n -vertex algebra U to another V is a linear map f such that $f(\mathbf{1}) = \mathbf{1}$ and

$$f(Y(u, \mathbf{x})v) = Y(f(u), \mathbf{x})f(v) \quad \text{for } u, v \in U. \quad (3.84)$$

Isomorphism and automorphism are defined in the obvious ways.

Just as in the case for an ordinary vertex algebra, from a routine proof we immediately have:

Lemma 3.21 *Let V be a G_n -vertex algebra and G a group of automorphisms of V . Set*

$$V^G = \{v \in V \mid g(v) = v \quad \text{for } g \in G\}. \quad (3.85)$$

Then V^G is a vertex subalgebra. \square

4 Vertex algebras of G_n -vertex operators

In this section, we shall continue the study of Section 2 on G_n -vertex operators on a vector space W . For any two (maybe the same) given mutually local G_n -vertex operators, we define a family of products parametrized by $\mathbf{m} \in \mathbf{Z}^n$. As our main results we prove that any maximal local space of G_n -vertex operators on W equipped with the defined products is a vertex algebra with W as a natural module and that any set of pair-wise mutually local G_n -vertex operators on W automatically generates a vertex algebra.

Now, we continue the discussion of Section 2 and we shall freely use the notations and conventions defined there. Let W be a vector space. In terms of maps $\iota_{\alpha, \mathbf{x}, \mathbf{y}}$, we introduce the following definition.

Definition 4.1 Let ψ and ϕ be mutually local G_n -vertex operators on W . For $\mathbf{m} \in \mathbf{Z}^n$, we define a formal series

$$(\psi_{\mathbf{m}}\phi)(\mathbf{y}) \in (\text{End } W)[[y_1, y_1^{-1}, \dots, y_n, y_n^{-1}]]$$

by

$$\begin{aligned} & (\psi_{\mathbf{m}}\phi)(\mathbf{y}) \\ = & \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{z}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \mathbf{z}^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) \right) \\ = & \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y})). \end{aligned} \quad (4.1)$$

Noting that if $\mathbf{m} \geq 0$, for every $\alpha \in (F_2)^n$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^{\mathbf{m}}) = (\mathbf{x} - \mathbf{y})^{\mathbf{m}},$$

we get

$$(\psi_{\mathbf{m}}\phi)(\mathbf{y}) = \text{Res}_{\mathbf{x}} (\mathbf{x} - \mathbf{y})^{\mathbf{m}} \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} (\psi(\mathbf{x}) \phi(\mathbf{y})). \quad (4.2)$$

(The sum is analogous to the usual commutator.)

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (F_2)^n$, $w \in W$. Consider $\text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) w)$. Let $k \in \mathbb{N}$ be such that (2.40) holds. Then

$$\begin{aligned} & \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) w) \\ = & \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}-k}) \left((\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x}) \phi(\mathbf{y}) w \right). \end{aligned} \quad (4.3)$$

For $1 \leq i \leq n$, if $\alpha_i = 0$, both $(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x}) \phi(\mathbf{y}) w$ and $\iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}-k})$ involves only finitely many negative powers of y_i (recall Remark 2.15), so does $\text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) w)$. If $\alpha_i = 1$, we have

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}-k}) = (-y_i + x_i)^{m_i-k} \prod_{j \neq i} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((x_j - y_j)^{m_j-k}).$$

Then

$$\text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}-k}) \left((\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x}) \phi(\mathbf{y}) w \right)$$

involves only finitely many negative powers of y_i , so does $\text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) w)$.

Thus, $\text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y}) w)$ always involves only finitely many negative powers of y_i 's. From (4.1) we have

$$(\psi_{\mathbf{m}} \phi)(\mathbf{y}) w \in W((\mathbf{y})) \quad \text{for } w \in W.$$

Thus, $(\psi_{\mathbf{m}} \phi)(\mathbf{y}) \in VO_{G_n}(W)$. Assume $m_i \geq k$ for some i . Using Corollary 2.20 we get

$$\begin{aligned} & \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y})) \\ &= \left(\prod_{j \neq i} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((x_j - y_j)^{m_j}) \right) (x_i - y_i)^{m_i} \iota_{\alpha, \mathbf{x}, \mathbf{y}} (\psi(\mathbf{x}) \phi(\mathbf{y})) \\ &= \left(\prod_{j \neq i} \iota_{\alpha, \mathbf{x}, \mathbf{y}} ((x_j - y_j)^{m_j}) \right) (x_i - y_i)^{m_i} \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}} (\psi(\mathbf{x}) \phi(\mathbf{y})) \\ &= \iota_{\sigma_i(\alpha), \mathbf{x}, \mathbf{y}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x}) \phi(\mathbf{y})). \end{aligned} \tag{4.4}$$

Then using the relation $(-1)^{|\alpha|} = -(-1)^{|\sigma_i(\alpha)|}$, we get $\psi_{\mathbf{m}} \phi = 0$. To summarize, we have:

Lemma 4.2 *Let ψ and ϕ be mutually local G_n -vertex operators on W . Then*

$$\psi_{\mathbf{m}} \phi \in VO_{G_n}(W) \quad \text{for } \mathbf{m} \in \mathbf{Z}^n. \tag{4.5}$$

Furthermore,

$$\psi_{\mathbf{m}} \phi = 0 \quad \text{if } m_i \geq k \quad \text{for some } i, \tag{4.6}$$

where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$ and k is a nonnegative integer such that (2.40) holds. \square

Set

$$Y(\psi, \mathbf{z}) \phi = \sum_{\mathbf{m} \in \mathbf{Z}^n} (\psi_{\mathbf{m}} \phi) \mathbf{z}^{\mathbf{m}-1}. \tag{4.7}$$

In terms of this generating function we have

$$(Y(\psi, \mathbf{z}) \phi)(\mathbf{y}) = \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \psi(\mathbf{x}) \phi(\mathbf{y}) \right). \tag{4.8}$$

Lemma 4.3 *Let ψ, ϕ be mutually local G_n -vertex operators on W and let $w \in W$. Then there exists $l \in \mathbb{N}$ (depending only on ψ and w) such that*

$$(\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{z}) \phi)(\mathbf{y}) w = (\mathbf{z} + \mathbf{y})^l \psi(\mathbf{z} + \mathbf{y}) \phi(\mathbf{y}) w. \tag{4.9}$$

In particular, we may take $l \geq 0$ such that $\mathbf{x}^l \psi(\mathbf{x}) w \in W[[[\mathbf{x}]]]$.

Proof. For any $l \in \mathbb{N}$, using (4.8) and delta function substitution property we have

$$\begin{aligned}
& (\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{z})\phi)(\mathbf{y})w \\
&= \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}}(\mathbf{y} + \mathbf{z})^l \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \psi(\mathbf{x})\phi(\mathbf{y})w \right) \\
&= \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \mathbf{x}^l \psi(\mathbf{x})\phi(\mathbf{y})w \right). \tag{4.10}
\end{aligned}$$

Let $l \geq 0$ be such that $\mathbf{x}^l \psi(\mathbf{x})w \in W[[\mathbf{x}]]$. Recall (2.50):

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\psi(\mathbf{x})\phi(\mathbf{y})) = \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\phi(\mathbf{y})\psi(\mathbf{x})).$$

Because for $m \in \mathbf{Z}$, $\iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x} - \mathbf{y})^m$ involves only nonnegative powers of x_i for i with $\alpha_i = 1$,

$$\iota_{\alpha, \mathbf{x}, \mathbf{y}}((\mathbf{x} - \mathbf{y})^m) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x}^l \phi(\mathbf{y})\psi(\mathbf{x})w)$$

involves only nonnegative powers of x_i for i with $\alpha_i = 1$. Then

$$\begin{aligned}
& \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left((\mathbf{x} - \mathbf{y})^m \mathbf{x}^l \psi(\mathbf{x})\phi(\mathbf{y})w \right) \\
&= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left((\mathbf{x} - \mathbf{y})^m \right) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x}^l \psi(\mathbf{x})\phi(\mathbf{y})w) \\
&= \text{Res}_{\mathbf{x}} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left((\mathbf{x} - \mathbf{y})^m \right) \iota_{\alpha, \mathbf{x}, \mathbf{y}}(\mathbf{x}^l \phi(\mathbf{y})\psi(\mathbf{x})w) \\
&= 0 \tag{4.11}
\end{aligned}$$

for all $m \in \mathbf{Z}$, $\alpha \neq (0, \dots, 0)$. Therefore

$$\begin{aligned}
& (\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{z})\phi)(\mathbf{y})w \\
&= \text{Res}_{\mathbf{x}} \mathbf{x}^l \mathbf{z}^{-1} \delta \left(\frac{\mathbf{x} - \mathbf{y}}{\mathbf{z}} \right) \psi(\mathbf{x})\phi(\mathbf{y})w \\
&= \text{Res}_{\mathbf{x}} (\mathbf{z} + \mathbf{y})^l \mathbf{x}^{-1} \delta \left(\frac{\mathbf{z} + \mathbf{y}}{\mathbf{x}} \right) \psi(\mathbf{x})\phi(\mathbf{y})w \\
&= (\mathbf{z} + \mathbf{y})^l \psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})w. \tag{4.12}
\end{aligned}$$

This completes the proof. \square

By observing both sides of (4.9) we see that the common quantity on the both sides lies in $W((\mathbf{y}, \mathbf{z}))$, so that

$$\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})w \in W((\mathbf{y}, \mathbf{z}))[(\mathbf{z} + \mathbf{y})^{-1}].$$

Then

$$\begin{aligned}
(\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{z})\phi)(\mathbf{y})w &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\mathbf{z} + \mathbf{y})^l \psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})w) \\
&= (\mathbf{y} + \mathbf{z})^l \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})w). \tag{4.13}
\end{aligned}$$

Thus

$$(Y(\psi, \mathbf{z})\phi)(\mathbf{y})w = \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})w). \tag{4.14}$$

Therefore, we have proved:

Corollary 4.4 *Let ψ and ϕ be mutually local G_n -vertex operators on W . Then*

$$(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) = \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})). \quad \square \quad (4.15)$$

Remark 4.5 We may use equation (4.15) alternatively to define $\psi_{\mathbf{m}}\phi$.

With Lemma 4.3, from Proposition 3.5 we immediately have:

Proposition 4.6 *Let ψ, ϕ be mutually local G_n -vertex operators on W . Then*

$$\begin{aligned} & \mathbf{y}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{z}}{\mathbf{y}}\right)(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \\ &= \sum_{\alpha \in (F_2)^n} (-1)^{|\alpha|} \iota_{\alpha, \mathbf{x}, \mathbf{y}} \left(\mathbf{z}^{-1}\delta\left(\frac{\mathbf{x}-\mathbf{y}}{\mathbf{z}}\right)\psi(\mathbf{x})\phi(\mathbf{y}) \right). \quad \square \end{aligned} \quad (4.16)$$

We shall need the following result:

Lemma 4.7 *Let ψ, ϕ and θ be pairwise mutually local G_n -vertex operators on W . Then there exists a nonnegative integer k such that*

$$(\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k\theta(\mathbf{x})(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) = (\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k(Y(\psi, \mathbf{z})\phi)(\mathbf{y})\theta(\mathbf{x}) \quad (4.17)$$

In particular, we may take $k \geq 0$ such that

$$(\mathbf{x} - \mathbf{y})^k[\theta(\mathbf{x}), \psi(\mathbf{y})] = 0, \quad (\mathbf{x} - \mathbf{y})^k[\theta(\mathbf{x}), \phi(\mathbf{y})] = 0. \quad (4.18)$$

Proof. Let $k \geq 0$ be such that (4.18) holds. Using Corollary 4.4, noticing that $\iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}$ is linear on $\mathbf{C}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ we obtain (4.17) as

$$\begin{aligned} & (\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k\theta(\mathbf{x})(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \\ &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k\theta(\mathbf{x})\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})) \\ &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})\theta(\mathbf{x})) \\ &= (\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k\iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y}))\theta(\mathbf{x}) \\ &= (\mathbf{x} - \mathbf{y} - \mathbf{z})^k(\mathbf{x} - \mathbf{y})^k(Y(\psi, \mathbf{z})\phi)(\mathbf{y})\theta(\mathbf{x}). \quad \square \end{aligned} \quad (4.19)$$

The following is the key result of this section (cf. [Li2]):

Proposition 4.8 *Let ψ, ϕ and θ be pairwise mutually local G_n -vertex operators on W . Then for any $\mathbf{m} \in \mathbf{Z}^n$, $\psi_{\mathbf{m}}\phi$ and θ are mutually local.*

Proof. Let $k \geq 0$ be such that (4.17) holds. For any fixed $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^n$, in view of Lemma 4.2, there exists $r \geq 0$ such that $z_i^{r+m_i} Y(\psi, \mathbf{z})\phi$ involves only nonnegative powers of z_i for $i = 1, \dots, n$, so that

$$\text{Res}_{\mathbf{z}}(\mathbf{x} - \mathbf{y} - \mathbf{z})^{k+r-1} \mathbf{z}^{\mathbf{m}+1} (\mathbf{x} - \mathbf{y})^k (Y(\psi, \mathbf{z})\phi)(\mathbf{y}) = 0 \quad (4.20)$$

for $\mathbf{i} = i_1, \dots, i_n \in \mathbb{N}^n$ with $i_j \geq r$ for some j . Then using Lemma 4.7 we obtain

$$\begin{aligned} & (\mathbf{x} - \mathbf{y})^{2k+r} \theta(\mathbf{x})(\psi_{\mathbf{m}}\phi)(\mathbf{y}) \\ &= \text{Res}_{\mathbf{z}} \mathbf{z}^{\mathbf{m}} (\mathbf{x} - \mathbf{y})^{2k+r} \theta(\mathbf{x})(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \\ &= \text{Res}_{\mathbf{z}} \sum_{\mathbf{i} \in \mathbb{N}^n} \binom{k+r}{1} (\mathbf{x} - \mathbf{y} - \mathbf{z})^{k+r-1} \mathbf{z}^{\mathbf{m}+1} (\mathbf{x} - \mathbf{y})^k \theta(\mathbf{x})(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \\ &= \text{Res}_{\mathbf{z}} \sum_{\mathbf{i} \in \mathbb{N}^n, i \leq r} \binom{k+r}{1} (\mathbf{x} - \mathbf{y} - \mathbf{z})^{k+r-1} \mathbf{z}^{\mathbf{m}+1} (\mathbf{x} - \mathbf{y})^k \theta(\mathbf{x})(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \\ &= \text{Res}_{\mathbf{z}} \sum_{\mathbf{i} \in \mathbb{N}^n, i \leq r} \binom{k+r}{1} (\mathbf{x} - \mathbf{y} - \mathbf{z})^{k+r-1} \mathbf{z}^{\mathbf{m}+1} (\mathbf{x} - \mathbf{y})^k (Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \theta(\mathbf{x}) \\ &= \text{Res}_{\mathbf{z}} \sum_{\mathbf{i} \in \mathbb{N}^n} \binom{k+r}{1} (\mathbf{x} - \mathbf{y} - \mathbf{z})^{k+r-1} \mathbf{z}^{\mathbf{m}+1} (\mathbf{x} - \mathbf{y})^k (Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \theta(\mathbf{x}) \\ &= \text{Res}_{\mathbf{z}} \mathbf{z}^{\mathbf{m}} (\mathbf{x} - \mathbf{y})^{2k+r} (Y(\psi, \mathbf{z})\phi)(\mathbf{y}) \theta(\mathbf{x}) \\ &= (\mathbf{x} - \mathbf{y})^{2k+r} (\psi_{\mathbf{m}}\phi)(\mathbf{y}) \theta(\mathbf{x}). \end{aligned} \quad (4.21)$$

This proves that $\psi_{\mathbf{m}}\phi$ and θ are mutually local. \square

Remark 4.9 Motivated by the ordinary case [Li2], for $\mathbf{m} \in \mathbf{Z}^n$, we consider the formal series defined by

$$(\psi_{[\mathbf{m}]}\phi)(\mathbf{y}) = \text{Res}_{\mathbf{x}} ((\mathbf{x} - \mathbf{y})^{\mathbf{m}} \psi(\mathbf{x})\phi(\mathbf{y}) - (-\mathbf{y} + \mathbf{x})^{\mathbf{m}} \phi(\mathbf{y})\psi(\mathbf{x})). \quad (4.22)$$

Clearly, $\psi_{[\mathbf{m}]}\phi \in VO_{G_n}(W)$. However, in general, $\psi_{[\mathbf{m}]}\phi$ will not be local with the old vertex operators ψ and ϕ . Notice that Corollary 2.20 is crucial in the proof of Proposition 4.8.

Proposition 4.10 *Let W be a vector space and $\mathbf{d} = (d_1, \dots, d_n)$ an n -tuple of endomorphisms of W . Let $VO_{G_n}(W, \mathbf{d})$ consist of G_n -vertex operators ψ on W such that*

$$[d_i, \psi(\mathbf{x})] = \partial_{x_i} \psi(\mathbf{x}) \quad \text{for } i = 1, \dots, n. \quad (4.23)$$

If $\psi, \phi \in VO_{G_n}(W, \mathbf{d})$ are mutually local, then

$$\psi_{\mathbf{m}}\phi \in VO_{G_n}(W, \mathbf{d}) \quad \text{for } \mathbf{m} \in \mathbf{Z}^n. \quad (4.24)$$

Proof. Using Corollary 4.4 we obtain

$$\begin{aligned}
& [d_i, (Y(\psi, \mathbf{z})\phi)(\mathbf{y})] \\
&= [d_i, \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y}))] \\
&= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}([d_i, \psi(\mathbf{z} + \mathbf{y})]\phi(\mathbf{y}) + \psi(\mathbf{z} + \mathbf{y})[d_i, \phi(\mathbf{y})]) \\
&= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\partial_{y_i}\psi(\mathbf{z} + \mathbf{y}))\phi(\mathbf{y}) + \psi(\mathbf{z} + \mathbf{y})\partial_{y_i}\phi(\mathbf{y})) \\
&= \partial_{y_i}\iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})) \\
&= \partial_{y_i}(Y(\psi, \mathbf{z})\phi)(\mathbf{y}). \tag{4.25}
\end{aligned}$$

In terms of components we have

$$[d_i, (\psi_{\mathbf{m}}\phi)(\mathbf{y})] = \partial_{y_i}(\psi_{\mathbf{m}}\phi)(\mathbf{y}) \tag{4.26}$$

for $i = 1, \dots, n$ and for $\mathbf{m} \in \mathbf{Z}^n$. This proves (4.24). \square

There are n linear operators on $VO_{G_n}(W)$:

$$D_i = \partial_{x_i} \left(= \frac{\partial}{\partial x_i} \right) \quad \text{for } i = 1, \dots, n. \tag{4.27}$$

Set

$$\mathbf{D} = (D_1, \dots, D_n) \in (\text{End } VO_{G_n}(W))^n. \tag{4.28}$$

Write

$$\mathbf{x}\mathbf{D} = x_1D_1 + \dots + x_nD_n. \tag{4.29}$$

Note that every vertex operator on W is mutually local with the identity operator 1 of W .

Lemma 4.11 *Let $\psi \in VO_{G_n}(W)$. Then*

$$(Y(\psi, \mathbf{z})1)(\mathbf{y}) = e^{\mathbf{z}\mathbf{D}}\psi(\mathbf{y}). \tag{4.30}$$

In particular, for $1 \leq i \leq n$,

$$D_i(\psi) = \text{Res}_{\mathbf{x}} \mathbf{x}^{-1} x_i^{-1} Y(\psi, \mathbf{x}) 1. \tag{4.31}$$

Proof. Using Corollary 4.4 we obtain (4.30) as

$$\begin{aligned}
(Y(\psi, \mathbf{z})1)(\mathbf{y}) &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})1(\mathbf{y})) \\
&= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})) \\
&= \psi(\mathbf{y} + \mathbf{z}) \\
&= e^{\mathbf{z}\mathbf{D}}\psi(\mathbf{y}). \quad \square \tag{4.32}
\end{aligned}$$

Definition 4.12 A local subspace S of $VO_{G_n}(W)$ is said to be *closed* if

$$\phi_{\mathbf{m}}\phi \in S \quad \text{for } \psi, \phi \in S, \mathbf{m} \in \mathbf{Z}^n. \quad (4.33)$$

By the term “*a vertex algebra of mutually local G_n -vertex operators on W* ” we mean a closed local subspace of $VO_{G_n}(W)$ that contains the identity operator 1.

In the following we shall justify the use of “vertex algebra” in the above definition by proving that a vertex algebra of mutually local G_n -vertex operators on W is a vertex algebra as defined in Section 3.

Since

$$\text{Res}_{\mathbf{x}} \mathbf{x}^{-1} x_i^{-1} (Y(\psi, \mathbf{x}) 1)(\mathbf{y}) = \partial_{x_i} \psi(\mathbf{x}) \quad (4.34)$$

for $i = 1, \dots, n$, any closed local subspace of $VO_{G_n}(W)$ is stable under the action of $D_i = \partial_{x_i}$ for $i = 1, \dots, n$.

As an immediate consequence of Proposition 4.8 we have:

Corollary 4.13 Any maximal local subspace (or subset) of $VO_{G_n}(W)$ is closed. \square

Remark 4.14 Let S be any local subset of $VO_{G_n}(W)$. Consider all the local subsets that contain S . It follows from Zorn’s lemma that there exists a maximal local subspace of $VO_{G_n}(W)$ that contains S . In view of Corollary 4.13, there exists a closed local subspace of $VO_{G_n}(W)$ that contains S and 1. Let $\langle S \rangle$ be the smallest closed local subspace of $VO_{G_n}(W)$ that contains S and the identity operator 1 of W . Then $\langle S \rangle$ is closed.

Furthermore, we have:

Proposition 4.15 Let V be a closed local subspace of $VO_{G_n}(W)$. Then

$$[D_i, Y(\psi, \mathbf{x})]\phi = \partial_{x_i} Y(\psi, \mathbf{x})\phi \quad \text{for } \psi, \phi \in V, i = 1, \dots, n. \quad (4.35)$$

Proof. Using Corollary 4.4 we obtain (4.35) as

$$\begin{aligned} & ([D_i, Y(\psi, \mathbf{z})]\phi)(\mathbf{y}) \\ &= D_i(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) - Y(\psi, \mathbf{z})D_i\phi(\mathbf{y}) \\ &= \partial_{y_i}(Y(\psi, \mathbf{z})\phi)(\mathbf{y}) - Y(\psi, \mathbf{z})D_i\phi(\mathbf{y}) \\ &= \partial_{y_i}\iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})) - \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\partial_{y_i}\phi(\mathbf{y})) \\ &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\partial_{y_i}\psi(\mathbf{z} + \mathbf{y}))\phi(\mathbf{y})) \\ &= \iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}((\partial_{z_i}\psi(\mathbf{z} + \mathbf{y}))\phi(\mathbf{y})) \\ &= \partial_{z_i}\iota_{\mathbf{0}, \mathbf{y}, \mathbf{z}}(\psi(\mathbf{z} + \mathbf{y})\phi(\mathbf{y})) \\ &= \partial_{z_i}(Y(\psi, \mathbf{z})\phi)(\mathbf{y}). \quad \square \end{aligned} \quad (4.36)$$

The following result gives the locality of adjoint vertex operators (vertex operators on the space of vertex operators):

Proposition 4.16 *Let V be a vertex algebra of mutually local multi-variable vertex operators on W . Then for $\psi, \phi \in V$, $Y(\psi, \mathbf{x})$ and $Y(\phi, \mathbf{y})$ are mutually local G_n -vertex operators on V .*

Proof. Given $\psi, \phi \in V$, let $\theta \in V$, $w \in W$. Let $l \in \mathbb{N}$ be such that

$$\mathbf{x}^l \psi(\mathbf{x}) w, \quad \mathbf{x}^l \phi(\mathbf{x}) w \in W[[\mathbf{x}]].$$

Then from Lemma 4.3 for any $\rho \in V$,

$$(\mathbf{x} + \mathbf{z})^l (Y(\psi, \mathbf{x}) \rho)(\mathbf{z}) w = (\mathbf{x} + \mathbf{z})^l \psi(\mathbf{x} + \mathbf{z}) \rho(\mathbf{z}) w, \quad (4.37)$$

$$(\mathbf{y} + \mathbf{z})^l (Y(\phi, \mathbf{y}) \theta)(\mathbf{z}) w = (\mathbf{y} + \mathbf{z})^l \phi(\mathbf{y} + \mathbf{z}) \theta(z) w. \quad (4.38)$$

Especially, this holds for $\rho = Y(\psi, \mathbf{y}) \theta \in V((\mathbf{y}))$. Then

$$\begin{aligned} & (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{x}) Y(\phi, \mathbf{y}) \theta)(\mathbf{z}) w \\ &= (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l \psi(\mathbf{x} + \mathbf{z}) (Y(\phi, \mathbf{y}) \theta)(\mathbf{z}) w \\ &= (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l \psi(\mathbf{x} + \mathbf{z}) \phi(\mathbf{y} + \mathbf{z}) \theta(z) w. \end{aligned} \quad (4.39)$$

Similarly, we have

$$(\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l (Y(\phi, \mathbf{y}) Y(\psi, \mathbf{x}) \theta)(\mathbf{z}) w = (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l \phi(\mathbf{y} + \mathbf{z}) \psi(\mathbf{x} + \mathbf{z}) \theta(z) w. \quad (4.40)$$

Let $k \in \mathbb{N}$ (only depending on ψ and ϕ) be such that

$$(\mathbf{x} - \mathbf{y})^k \psi(\mathbf{x}) \phi(\mathbf{y}) = (\mathbf{x} - \mathbf{y})^k \phi(\mathbf{y}) \psi(\mathbf{x}). \quad (4.41)$$

Using (4.39), (4.40) and (4.41) we get

$$\begin{aligned} & (\mathbf{x} - \mathbf{y})^k (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l (Y(\psi, \mathbf{x}) Y(\phi, \mathbf{y}) \theta)(\mathbf{z}) w \\ &= (\mathbf{x} - \mathbf{y})^k (\mathbf{x} + \mathbf{z})^l (\mathbf{y} + \mathbf{z})^l (Y(\phi, \mathbf{y}) Y(\psi, \mathbf{x}) \theta)(\mathbf{z}) w, \end{aligned} \quad (4.42)$$

noting that $(\mathbf{x} - \mathbf{y})^k = ((\mathbf{x} + \mathbf{z}) - (\mathbf{y} + \mathbf{z}))^k$. Because of the lower truncation condition, we may multiply both sides by $(\mathbf{z} + \mathbf{x})^{-l} (\mathbf{z} + \mathbf{y})^{-l}$ to get

$$(\mathbf{x} - \mathbf{y})^k (Y(\psi, \mathbf{x}) Y(\phi, \mathbf{y}) \theta)(\mathbf{z}) w = (\mathbf{x} - \mathbf{y})^k (Y(\phi, \mathbf{y}) Y(\psi, \mathbf{x}) \theta)(\mathbf{z}) w. \quad (4.43)$$

Note that k does not depend on w . The assertion is proved. \square

Combining Lemma 4.11, Propositions 4.15 and 4.16 with Theorem 3.7 and also using Proposition 4.6 we immediately have (cf. [Li2]):

Theorem 4.17 *Let W be a vector space. Then any closed local subspace of $VO_{G_n}(W)$ is a vertex algebra with W as a natural module. In particular, any maximal local subspace of $VO_{G_n}(W)$ is a vertex algebra with W as a natural module. \square*

Let S be a set of pair-wise mutually local G_n -vertex operators on W . Recall from Remark 4.14 that $\langle S \rangle$ is the smallest closed local subspace of $VO_{G_n}(W)$ that contains S and the identity operator 1 of W . In view of Theorem 4.17, $\langle S \rangle$ is a vertex algebra with W as a natural module. Therefore we have (cf. [Li2]):

Theorem 4.18 *Let W be a vector space. Then any set of pair-wise mutually local G_n -vertex operators on W generates a canonical vertex algebra with W as a natural module. \square*

Combining Theorem 4.18 with Proposition 4.10 we immediately have:

Proposition 4.19 *Let W be a vector space and let $\mathbf{d} = (d_1, \dots, d_n)$ be an n -tuple of endomorphisms of W . Then any set of pair-wise mutually local G_n -vertex operators in $VO_{G_n}(W, \mathbf{d})$ generates a canonical vertex algebra V with W as a natural module such that*

$$[d_i, Y(v, \mathbf{x})] = \partial_{x_i} Y(v, \mathbf{x}) \quad \text{for } i = 1, \dots, n, v \in V. \quad (4.44)$$

The following is an analogue of a theorem of [FKRW] (see also [MP1-2], [X]):

Proposition 4.20 *Let V be a vector space, let $\mathbf{d} = (d_1, \dots, d_n)$ be an n -tuple of endomorphisms of V , $\mathbf{1}$ a vector in V and U a subspace of V equipped with a linear map Y from U to $VO_{G_n}(V)$ such that*

(C1) V is generated from $\mathbf{1}$ by all $u_{\mathbf{m}}$ for $u \in U$, $\mathbf{m} \in \mathbf{Z}^n$, where $Y(u, \mathbf{x}) = \sum_{\mathbf{m}} u_{\mathbf{m}} \mathbf{x}^{-\mathbf{m}-1}$.

(C2) $Y(u, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]]$ and $\lim_{\mathbf{x} \rightarrow 0} Y(u, \mathbf{x})\mathbf{1} = u$ for $u \in U$.

(C3) For $u \in U$, $1 \leq i \leq n$, $[d_i, Y(u, \mathbf{x})] = \partial_{x_i} Y(u, \mathbf{x})$.

(C4) For $u, v \in U$, $Y(u, \mathbf{x})$ and $Y(v, \mathbf{x})$ are mutually local.

(C5) $d_i(\mathbf{1}) = 0$ for $i = 1, \dots, n$.

Then there is a unique G_n -vertex algebra structure on V extending the linear map Y on U with $\mathbf{1}$ being the vacuum vector.

Proof. Due to (C1), the uniqueness follows immediately from the iterate formula (3.25) and induction. Set

$$S = \{Y(u, \mathbf{x}) \mid u \in U\}. \quad (4.45)$$

By (C3)-(C4), S is a local set of $VO_{G_n}(V, \mathbf{d})$. In view of Proposition 4.19, S generates a G_n -vertex algebra \tilde{V} with V as a module such that

$$[d_i, Y(\psi, \mathbf{x})] = \partial_{x_i} Y(\psi, \mathbf{x}) \quad \text{for } i = 1, \dots, n, \psi \in \tilde{V}. \quad (4.46)$$

With (C2), in view of Lemma 3.15, we have

$$Y(\psi, \mathbf{x})\mathbf{1} \in V[[\mathbf{x}]] \quad \text{for } \psi \in \tilde{V}. \quad (4.47)$$

and we have a \tilde{V} -homomorphism f from \tilde{V} to V defined by

$$f(\psi) = \lim_{\mathbf{x} \rightarrow 0} Y(\psi, \mathbf{x}) \mathbf{1} = \lim_{\mathbf{x} \rightarrow 0} \psi(\mathbf{x}) \mathbf{1} \quad \text{for } \psi \in \tilde{V}. \quad (4.48)$$

Note that we use $\mathbf{1}$ (the identity map on V) for the vacuum vector of \tilde{V} and $\mathbf{1}$ is the given element of V . Furthermore, by Proposition 3.16, f is an isomorphism. Then V has a vertex algebra structure transported from \tilde{V} through the linear isomorphism f . By (C2) we have $f(Y(u, \mathbf{x})) = u$ for $u \in U$. Therefore, the vertex algebra structure on V extends the linear map Y on U . \square

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